Mathématiques - Informatique

ALICE

Geometry and Light

Bruno Lévy
ALICE Géométrie & Lumière
CENTRE INRIA Nancy Grand-Est

OVERVIEW

Part. 1. Research axes, Evolutions, Applications

Part. 2. From Graphics to Fabrication

Part. 3. From Geometry Processing to Applied Math.

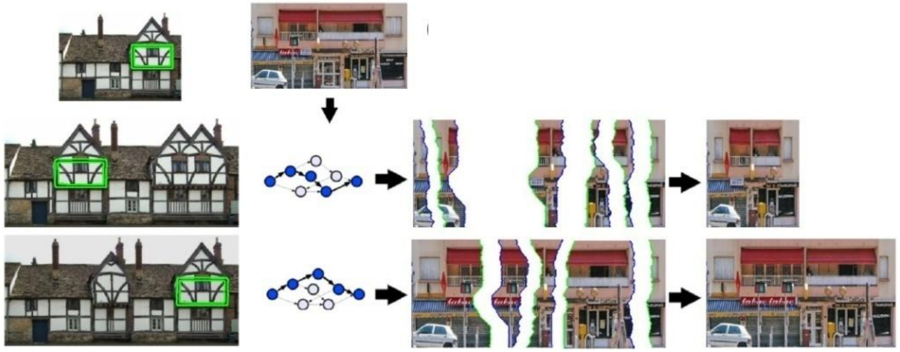
Part. 4. Future Works

1

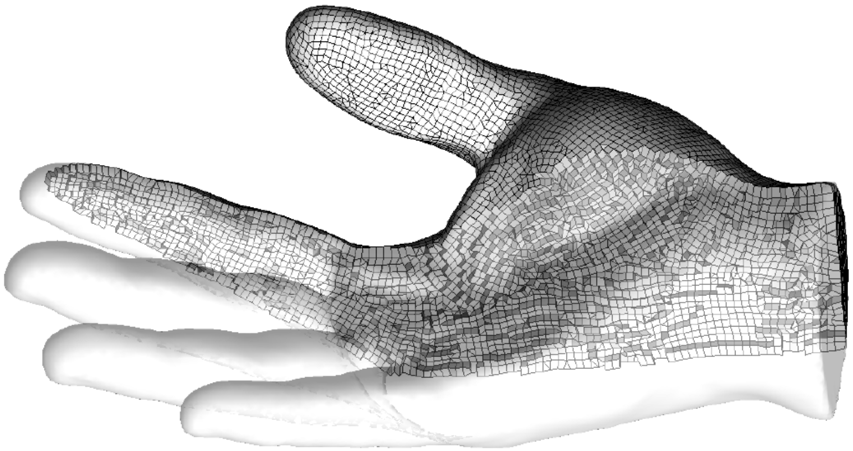
Research Axes
Evolution
Application Domains

Part. 1. Research Axes 2006 - 2012

Computer Graphics / Automatic Content Creation



Geometry Processing



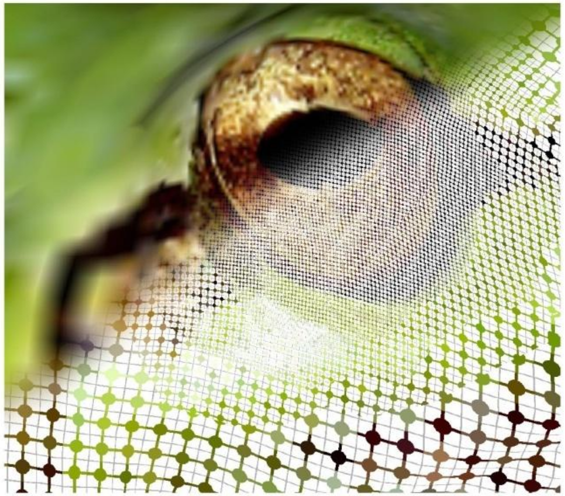
$$
\begin{aligned}
F_{L_p}^{T} &= \int_{T} \|\mathbf{M}_T(\mathbf{y}-\mathbf{x}_0)\|_p^p \, d\mathbf{y} \\
&= \frac{|T|}{\binom{n+p}{n}} \sum_{\alpha+\beta+\gamma=p} \overline{\mathbf{U}_1^{\alpha} * \mathbf{U}_2^{\beta} * \mathbf{U}_3^{\gamma}}
\end{aligned}
$$

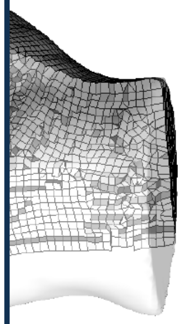
$$
\text{where:} \quad
\begin{cases}
\mathbf{U_i} = \mathbf{M}_T(\mathbf{C}_i - \mathbf{x}_0) \\
\mathbf{V}_1 * \mathbf{V}_2 = [x_1 x_2, y_1 y_2, z_1 z_2]^t \\
\mathbf{V}^{\alpha} = \mathbf{V} * \mathbf{V} * \ldots * \mathbf{V} (\alpha \text{ times}) \\
\overline{\mathbf{V}} = x + y + z
\end{cases}
$$

Part. 1. Research Axes 2006 - 2012

Computer Graphics / Automatic Content Creation

Geometry Processing





Texture Mapping

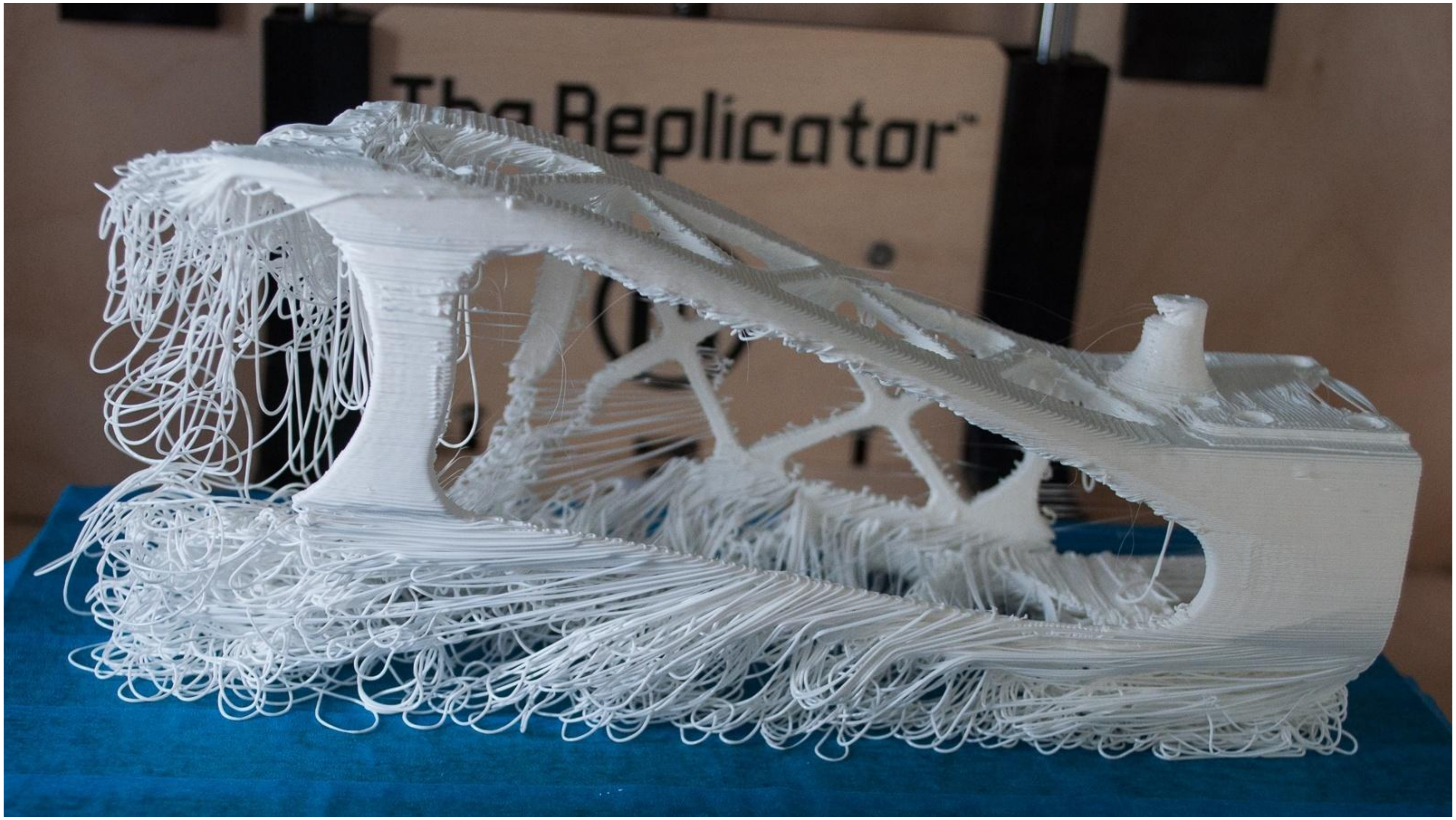
$$\text{where: } \begin{cases} \mathbf{U_i} & = \mathbf{M}_T(\mathbf{C}_i - \mathbf{x}_0) \\ \mathbf{V}_1 * \mathbf{V}_2 & = [x_1x_2, y_1y_2, z_1z_2]^t \\ \mathbf{V}^\alpha & = \mathbf{V} * \mathbf{V} * \ldots * \mathbf{V}(\alpha \text{ times}) \\ \overline{\mathbf{V}} & = x + y + z \end{cases}$$

Part. 1. Research Axes 2013 - 2016

Into reality

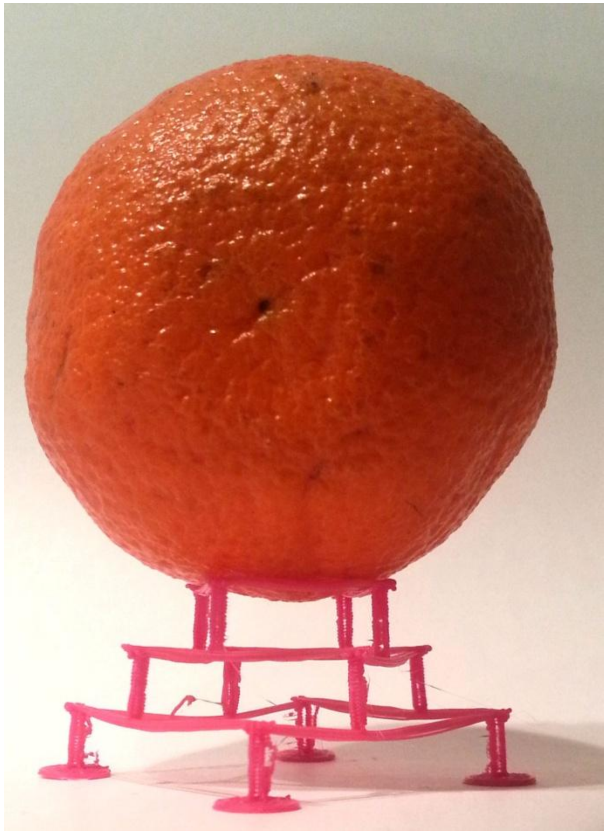
Poppy – Inria project Flowers

Part. 1. Research Axes 2013 - 2016



Print your own “Poppy Robot” at home … not that easy !!!

Part. 1. Research Axes 2013 - 2016



Print a “scaffold” with the object

Part. 1. Research Axes 2013 - 2016

Into reality

Poppy – Inria project Flowers

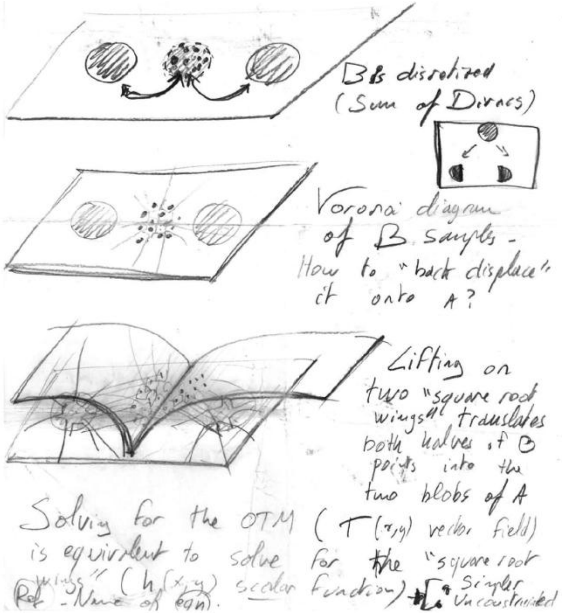
Make it easy for everybody (“it” = object modeling, 3d printing …)

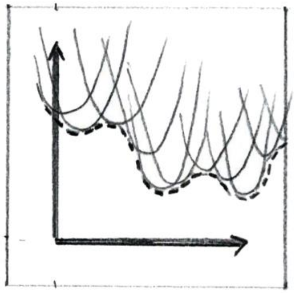
Part. 1. Research Axes 2013 - 2016

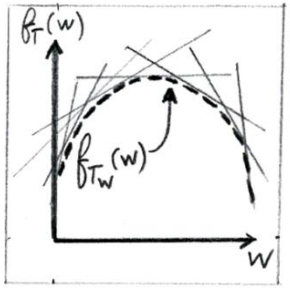
Into reality

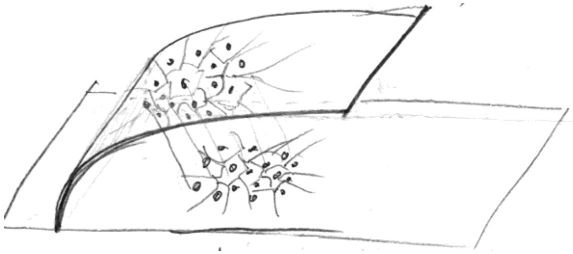
Poppy – Inria project Flowers

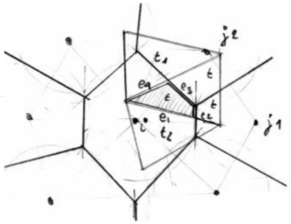
Into abstraction









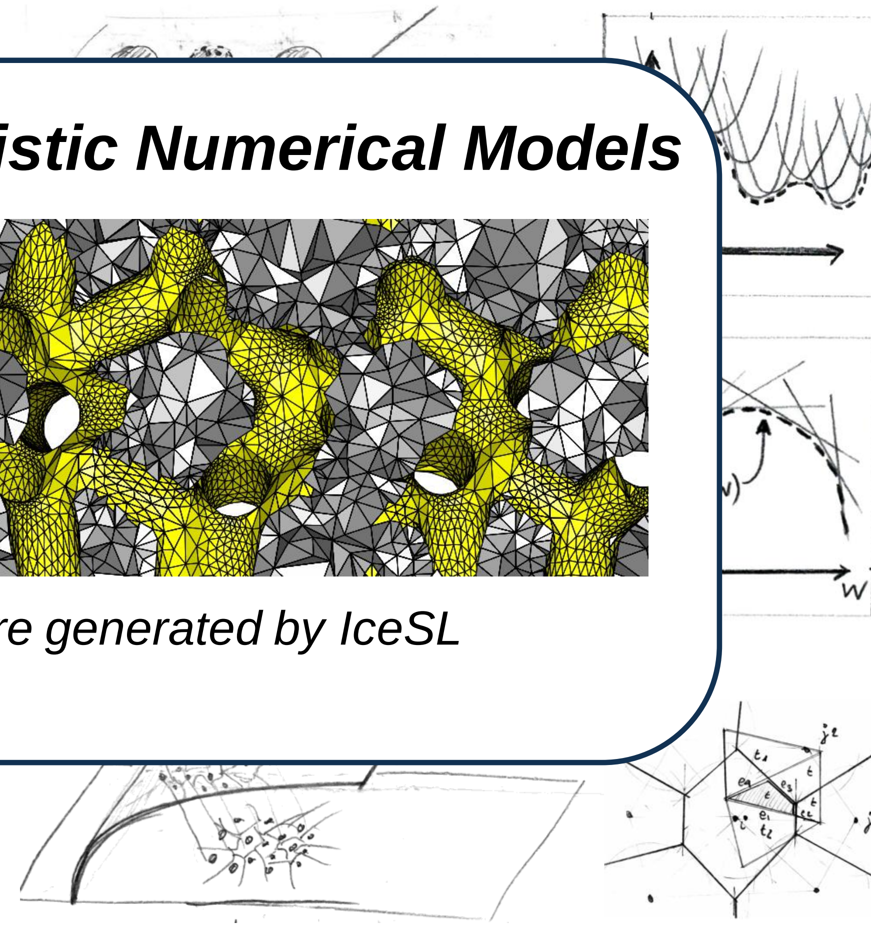


Part. 1. Research Axes 2013 - 2016

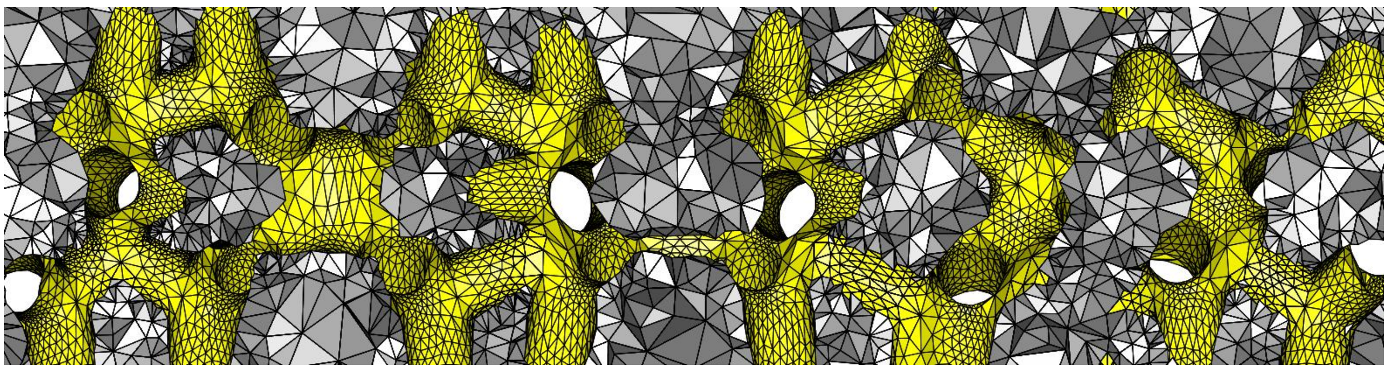
Into reality

Poppy – Inria project Flowers

Into abstraction



Simulating Reality – Realistic Numerical Models



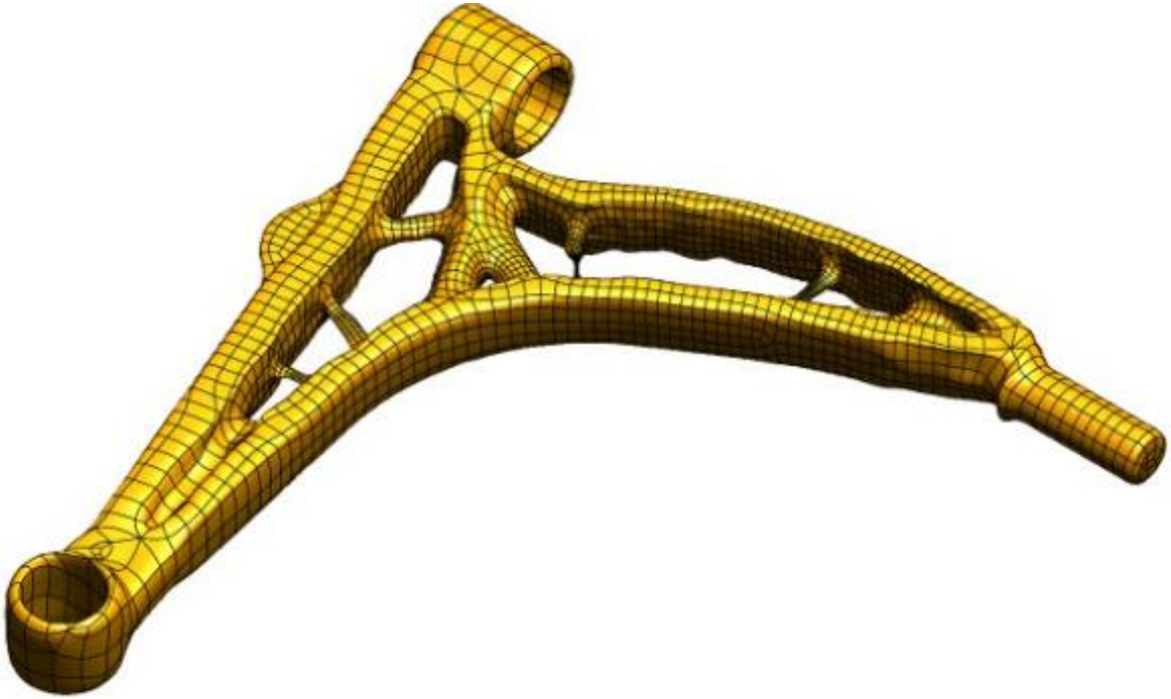
3D mesh of a microstructure generated by IceSL

Part. 1. Application Domains

Inserting Shape Optimization into the loop

Part. 1. Application Domains

Inserting Shape Optimization into the loop



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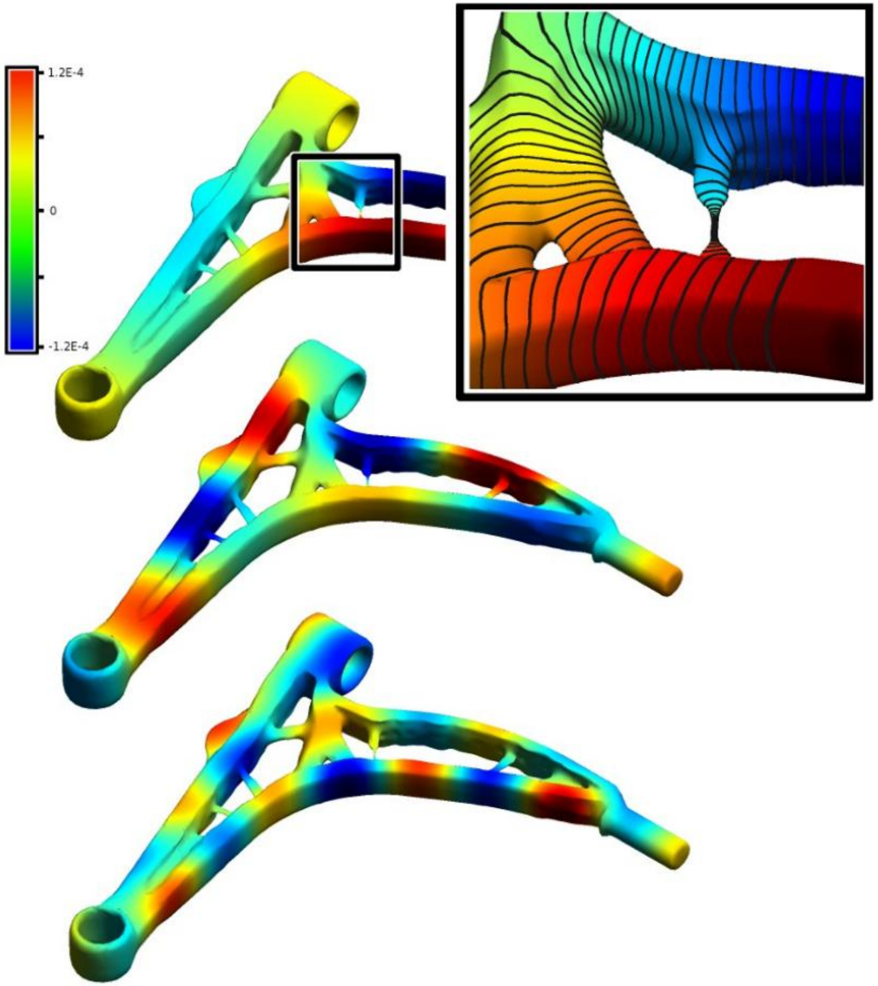
Inserting Shape Optimization into the loop



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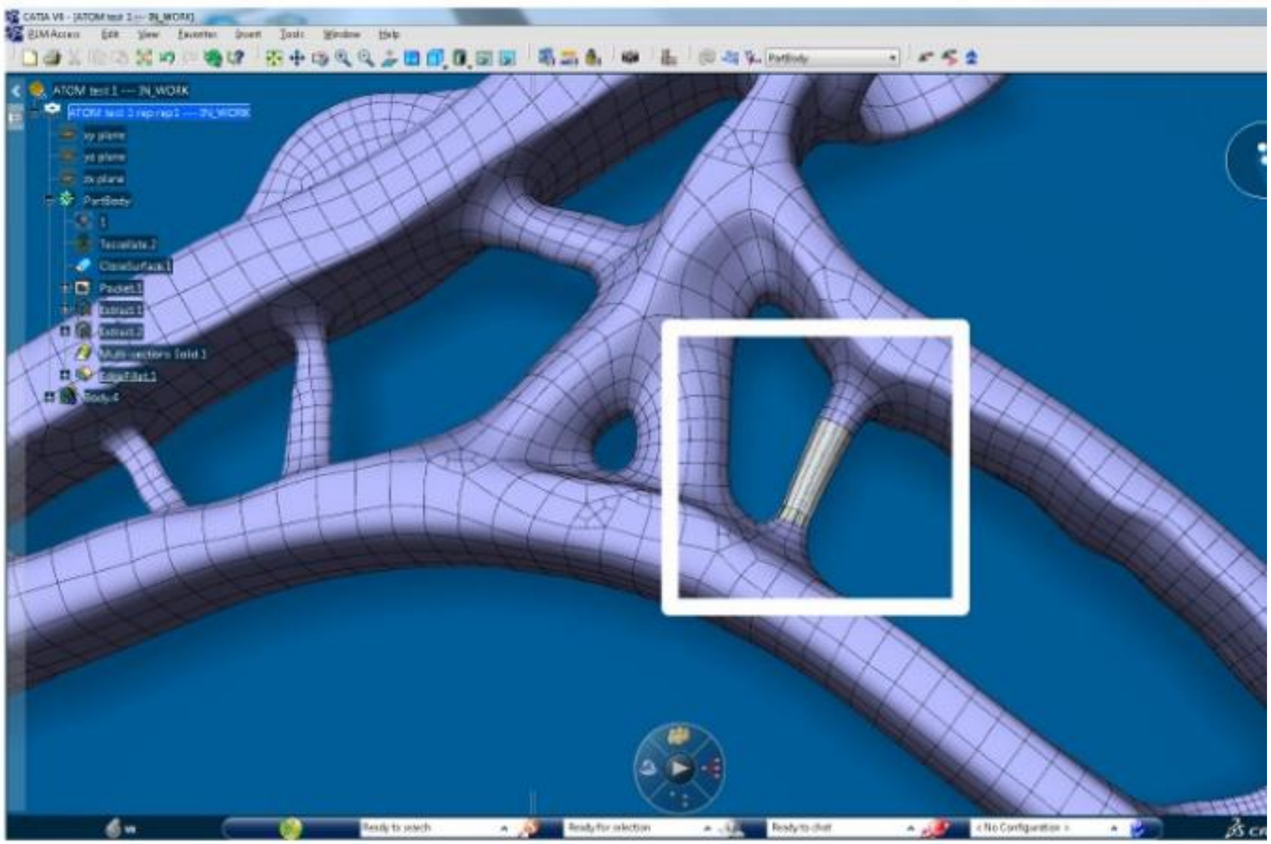
Inserting Shape Optimization into the loop

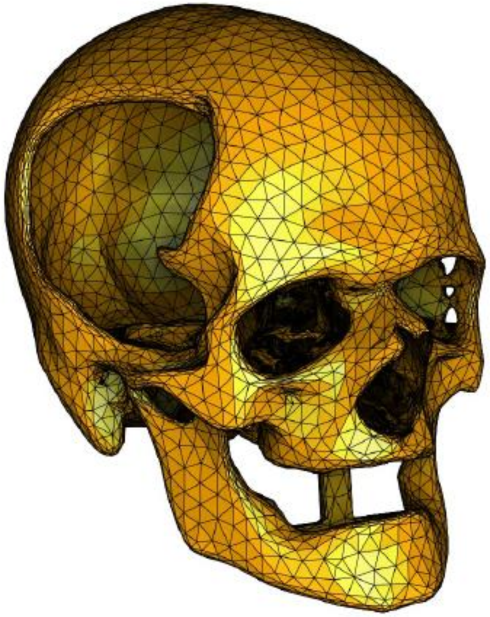


Part. 1. Application Domains

Inserting Shape Optimization into the loop

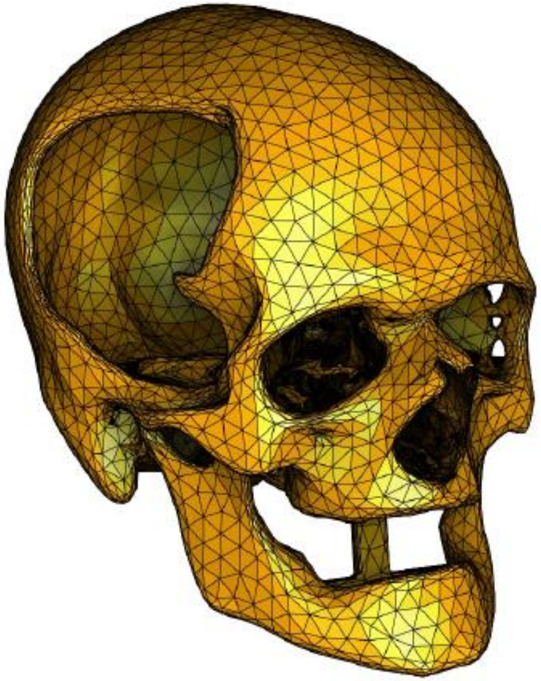
Part. 1. Application Domains

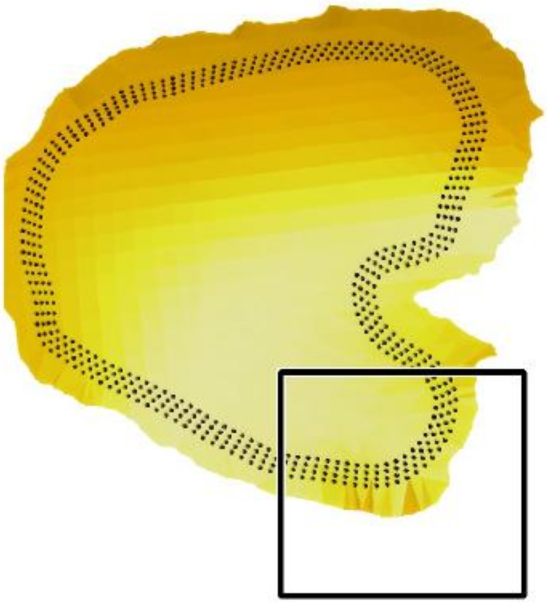
Reparative Surgery



Part. 1. Application Domains

Reparative Surgery





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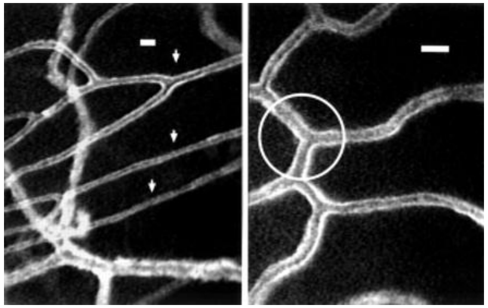
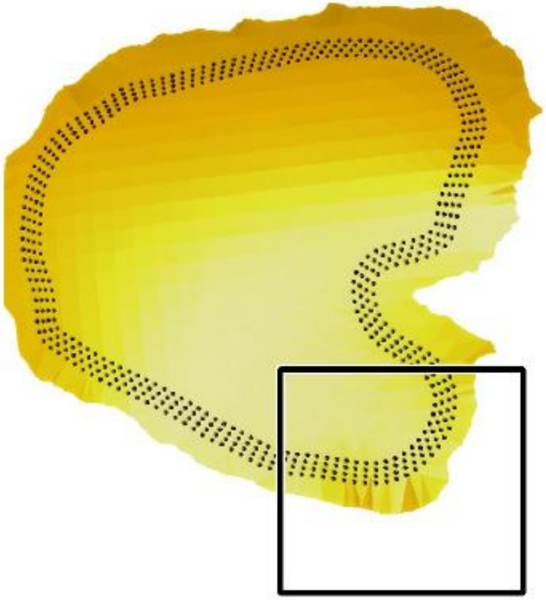
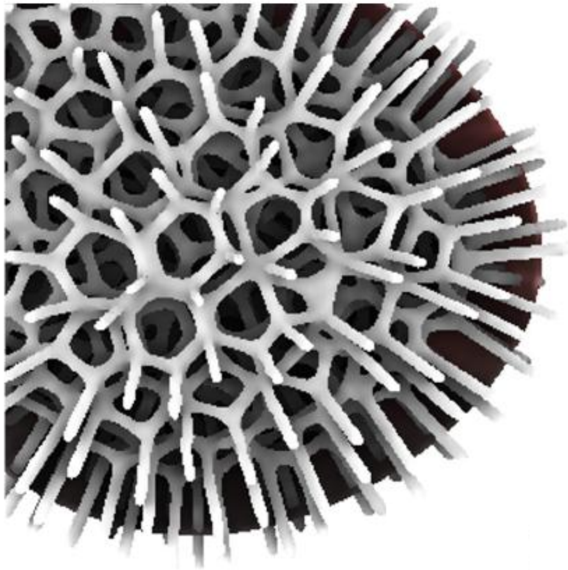
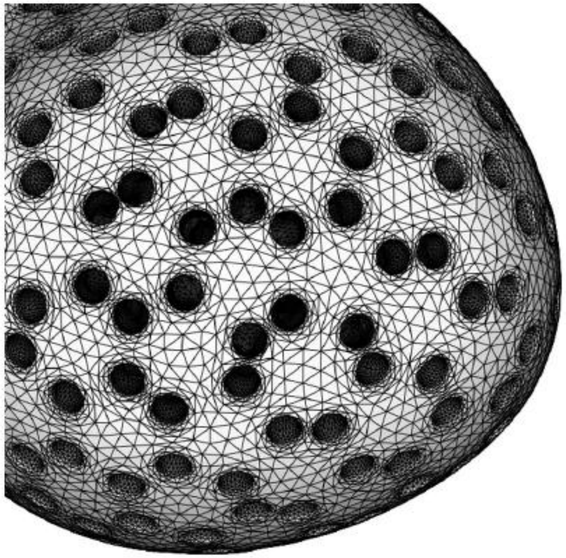
Reparative Surgery



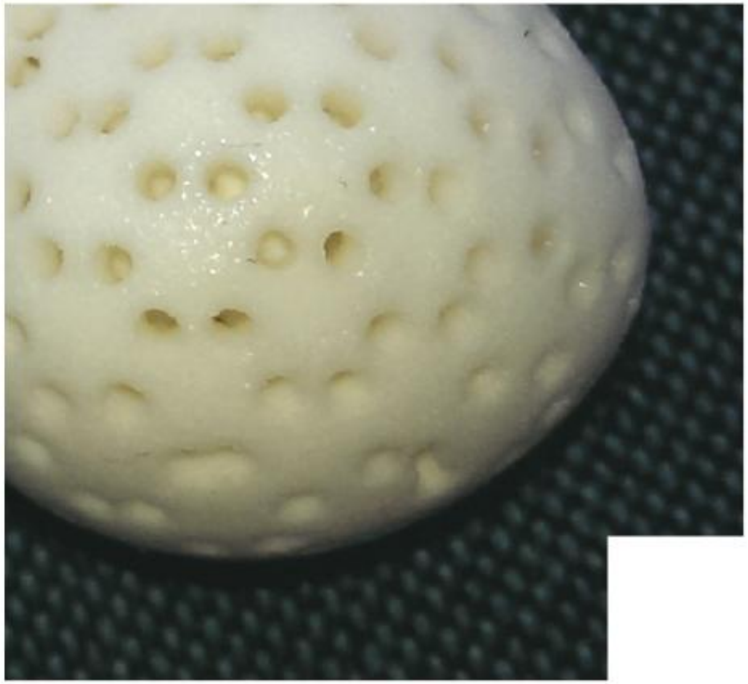






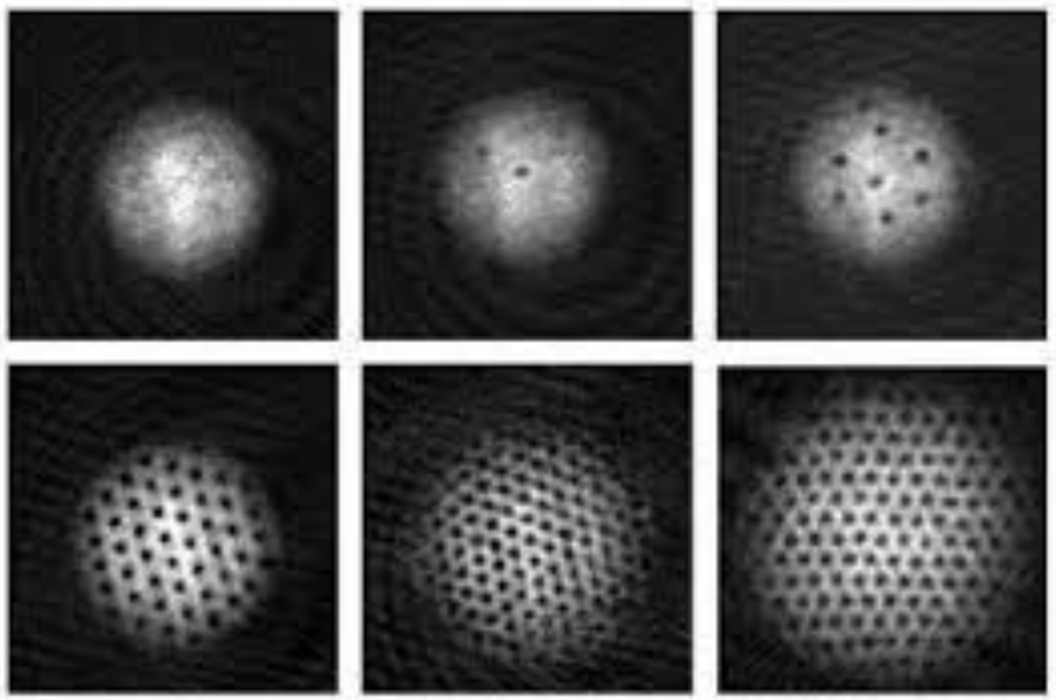
Part. 1. Application Domains

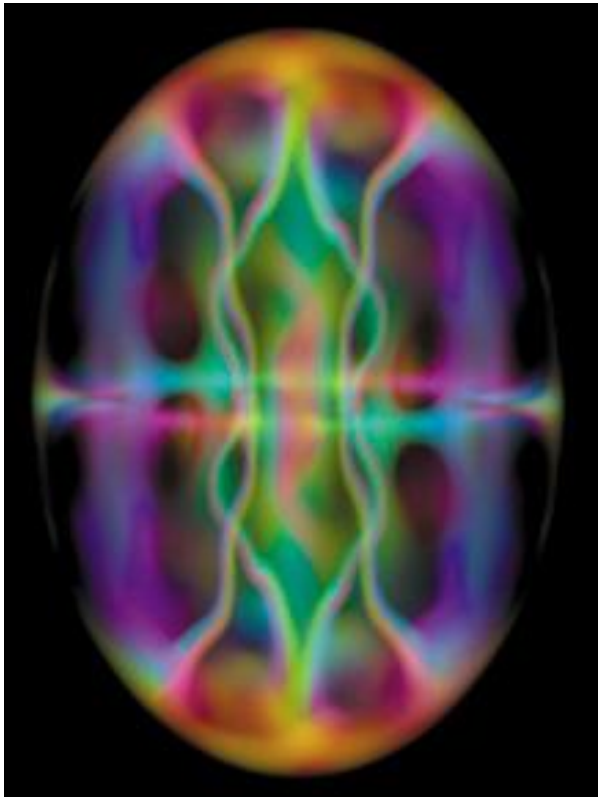
Reparative Surgery



Part. 1. Application Domains

Computational Physics





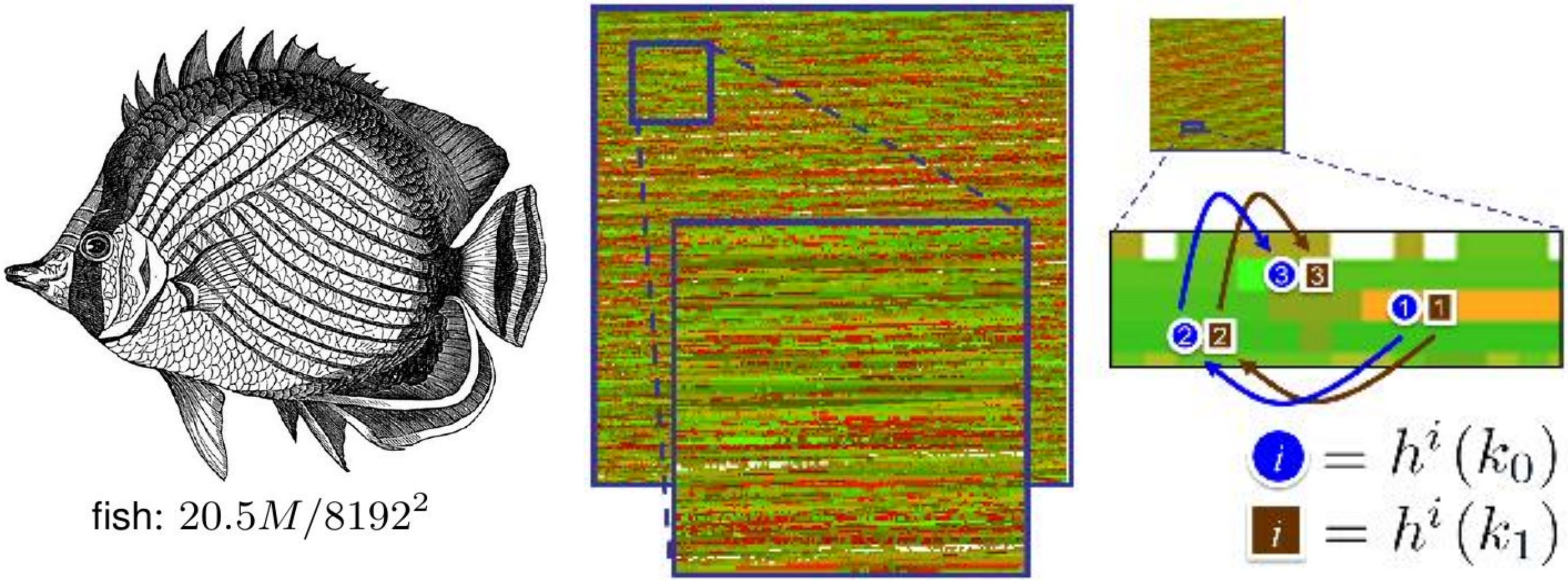
Bose-Einstein Condensate

ANR BECASIM – cooperation with physicists and mathematicians

2

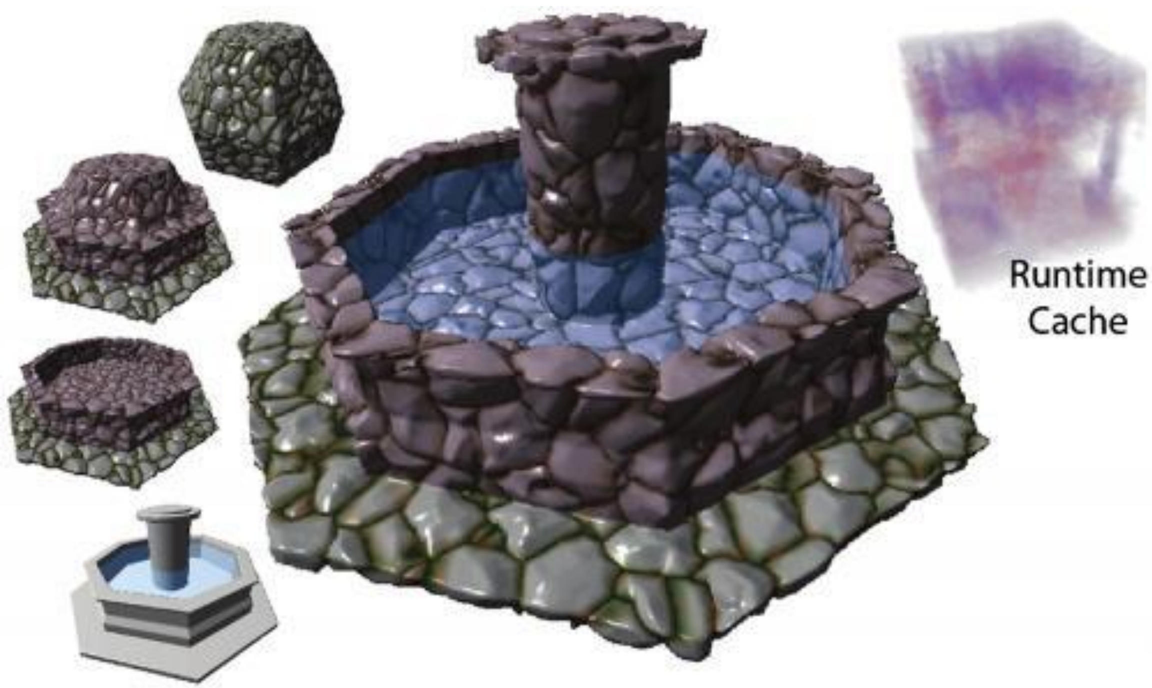
From Graphics to Fabrication

Part. 2 From Graphics to Fabrication



Coherent Parallel Hashing
Garcia, Lefebvre, Hornus, Lasram
SIGGRAPH Asia 2011

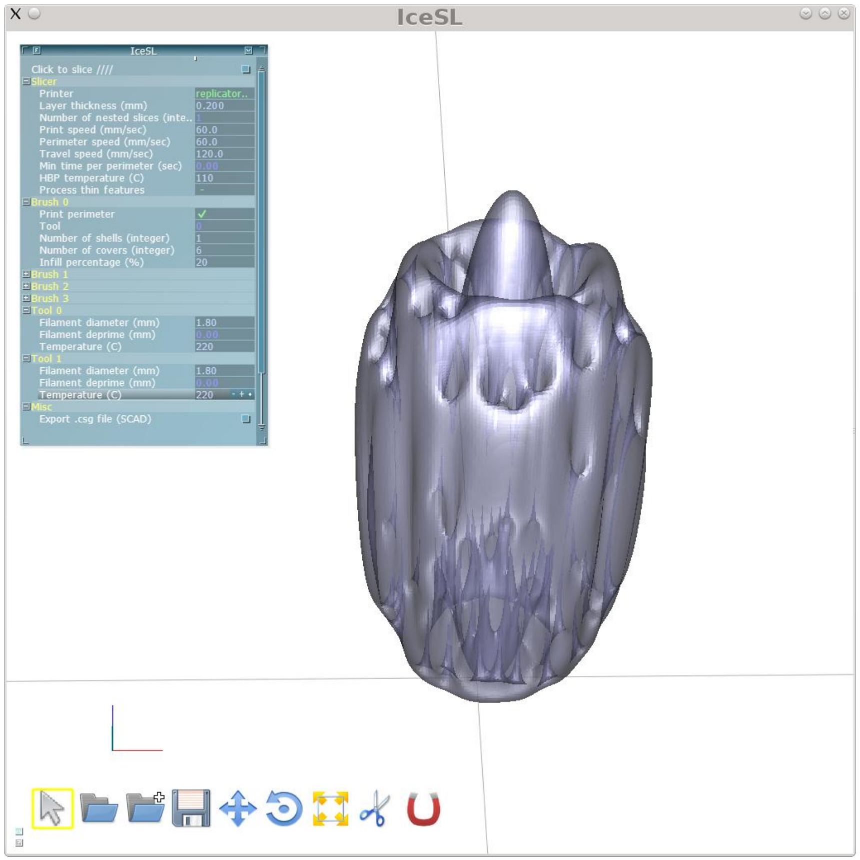
Part. 2 From Graphics to Fabrication

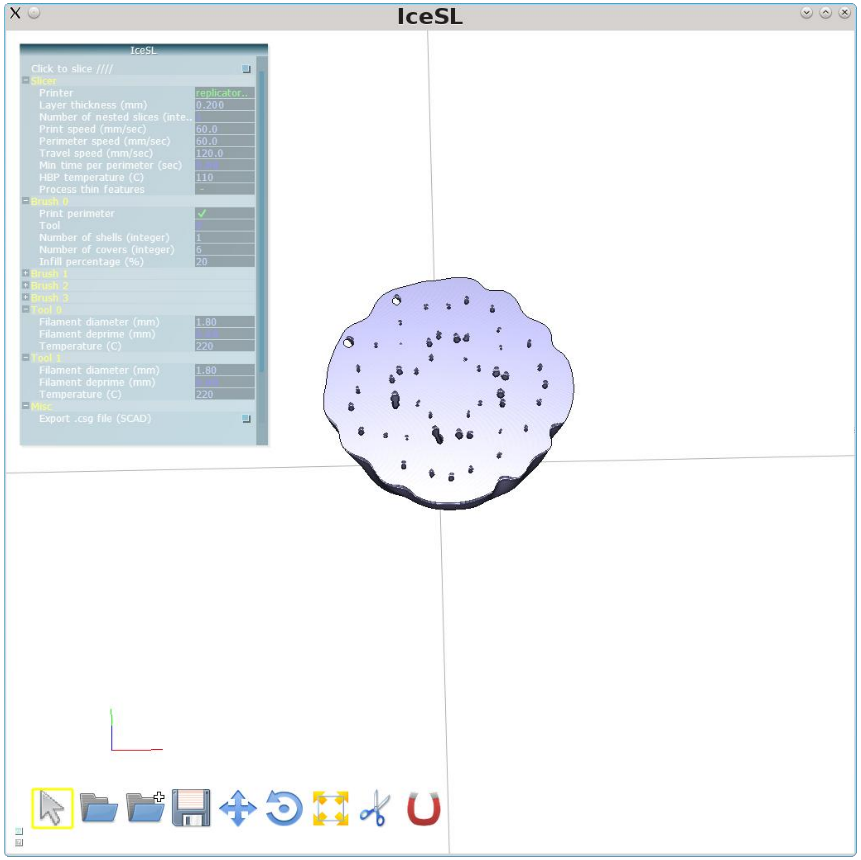


A runtime cache for interactive procedural modeling
Reiner, Lefebvre, Diener, Garcia, Jobard, Dachsbacher
SMI 2012

Part. 2 From Graphics to Fabrication

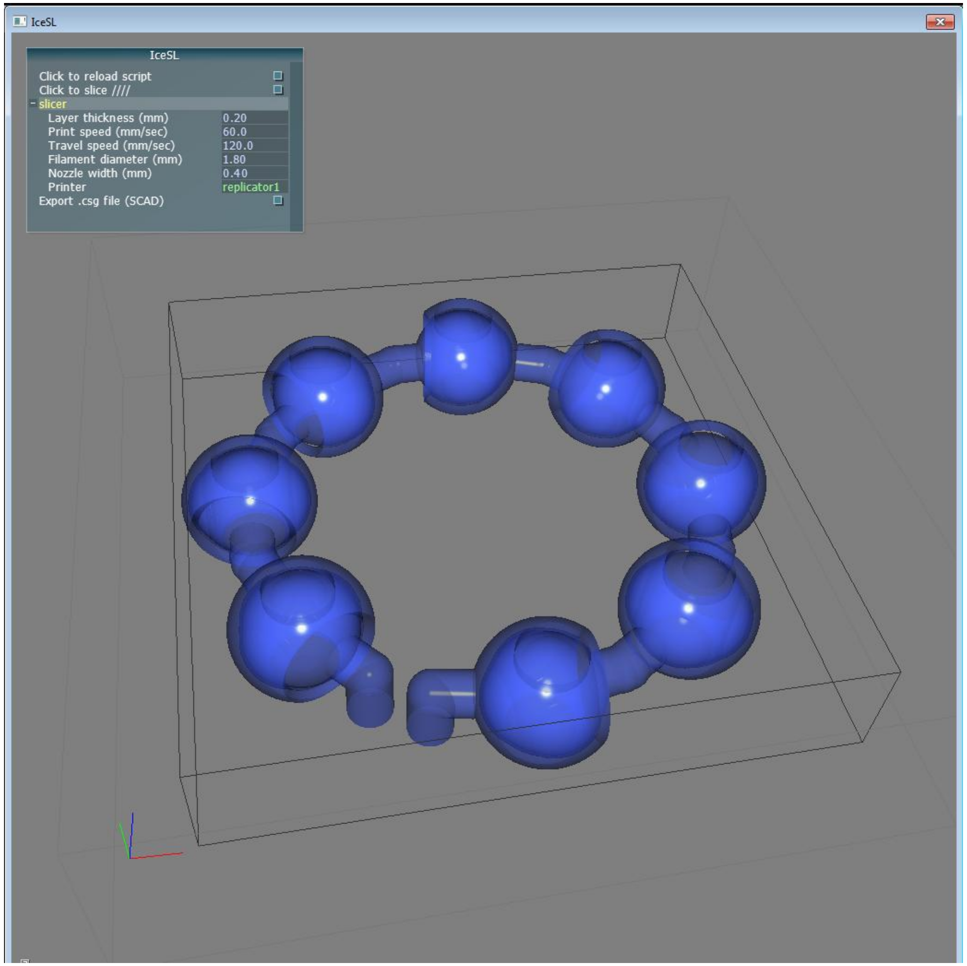
Visualization of Bose-Einstein condensates with IceSL



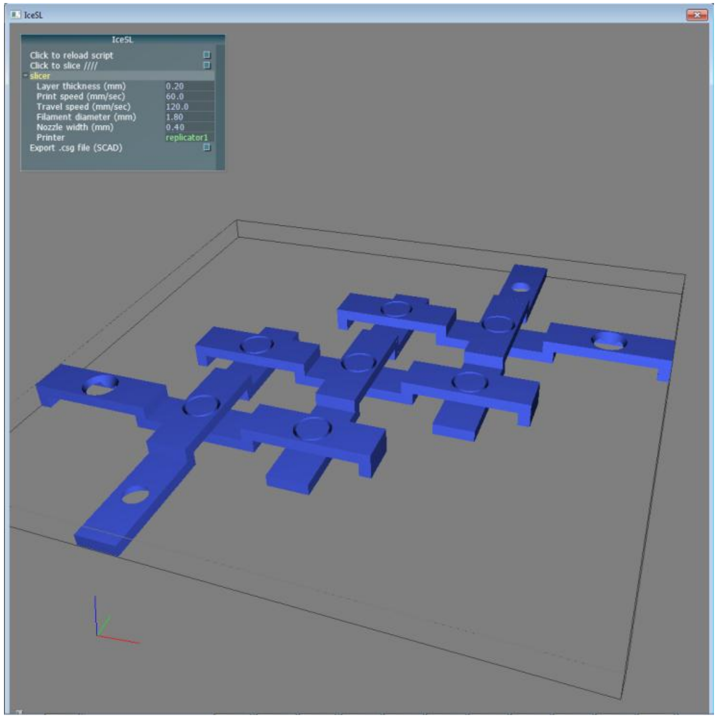


ANR BECASIM – cooperation with physicists and mathematicians

Part. 2 From Graphics to Fabrication



Part. 2 From Graphics to Fabrication



Part. 2 From Graphics to Fabrication

Make it stand, Prevost, Whiting, Lefebvre, Sorkine, SIGGRAPH 2012

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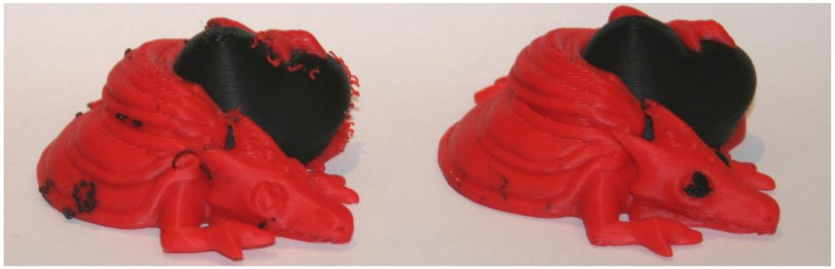
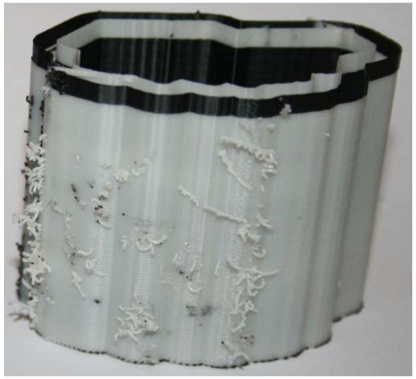
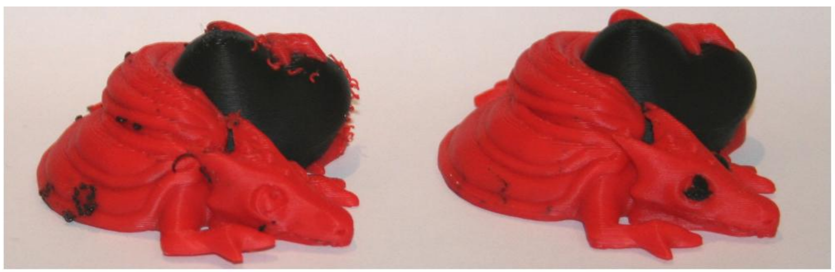


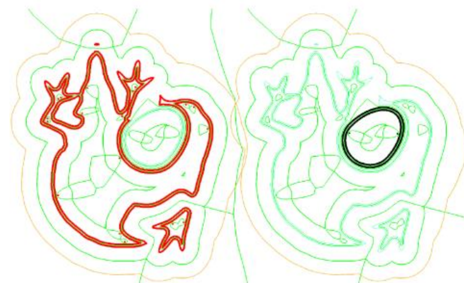


Clean Color, Hergel, Lefebvre, Eurographics 2014

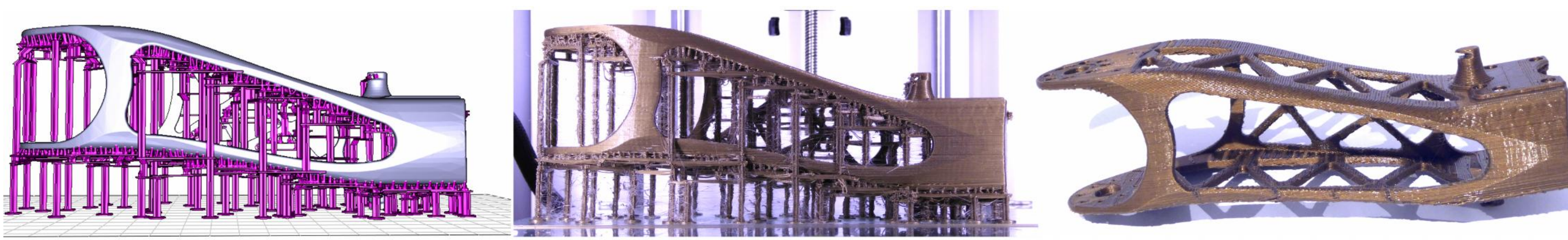
Part. 2 From Graphics to Fabrication

Make it stand, Prevost, Whiting, Lefebvre, Sorkine, SIGGRAPH 2012





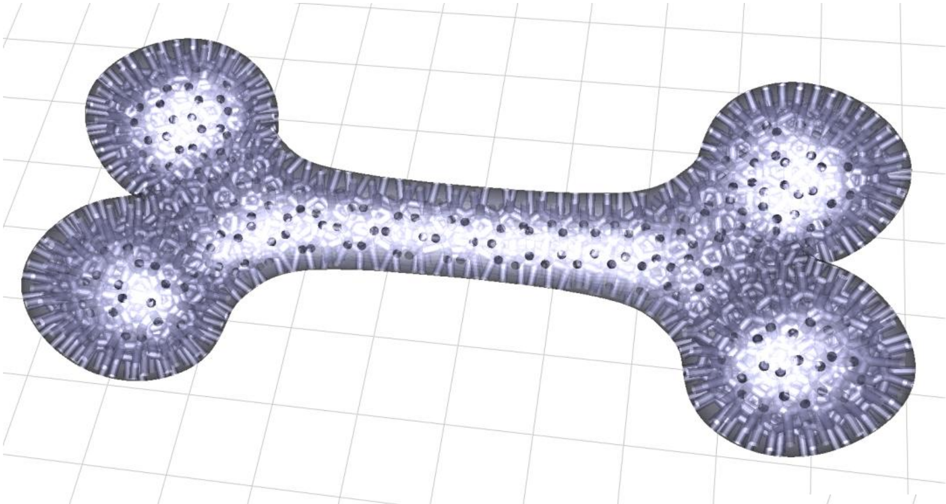
Clean Color, Hergel, Lefebvre, Eurographics 2014

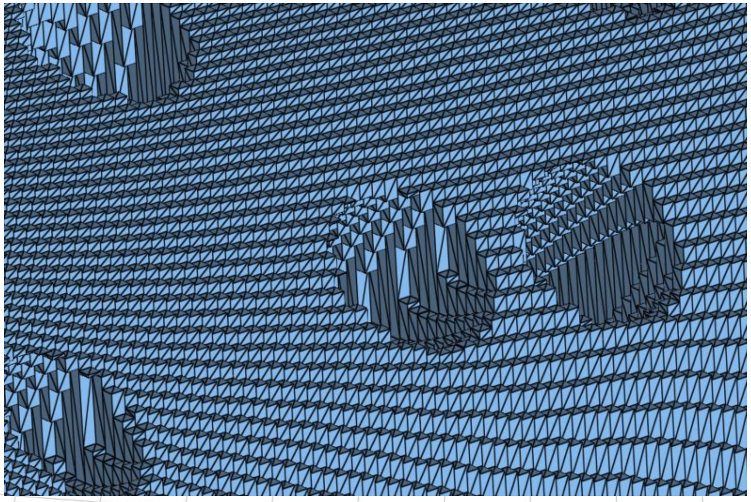


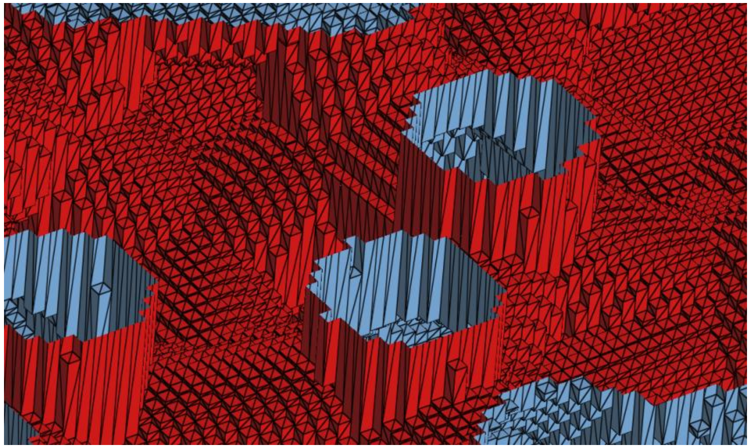
Bridge the gap, Dumas, Hergel, Lefebvre, SIGGRAPH 2014

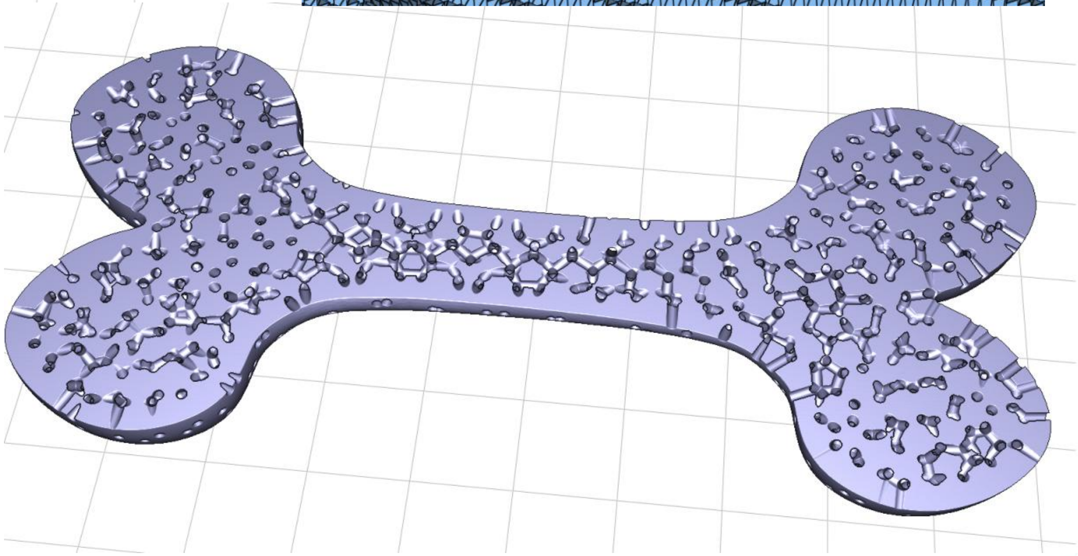
Part. 2 From Graphics to Fabrication

Reparative Surgery – toy example





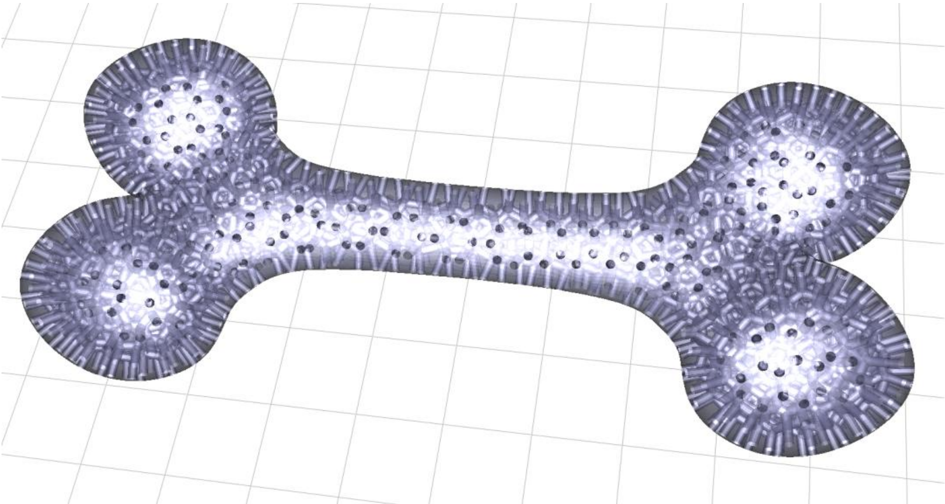




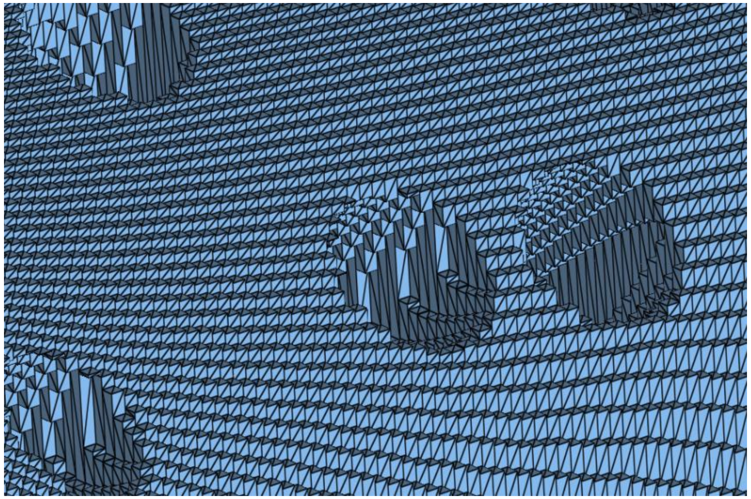
3

From Geometry Processing to Applied Mathematics

Part. 3 From Geometry Processing to Applied Math.



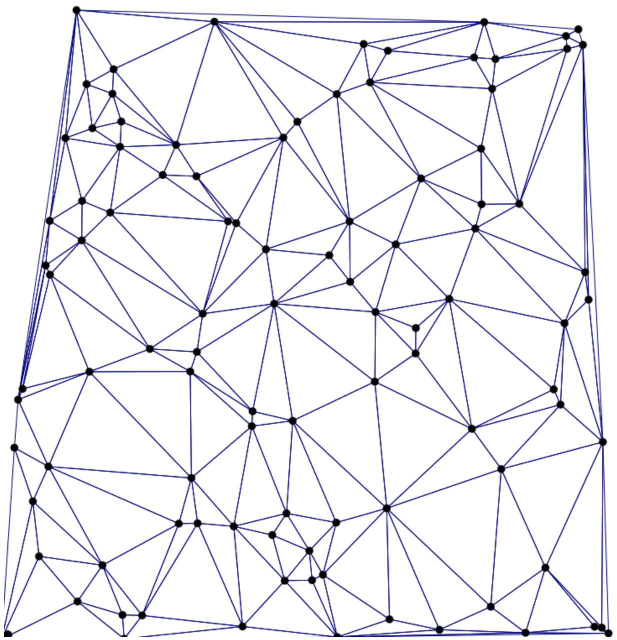
Exotic representation (Dexels)



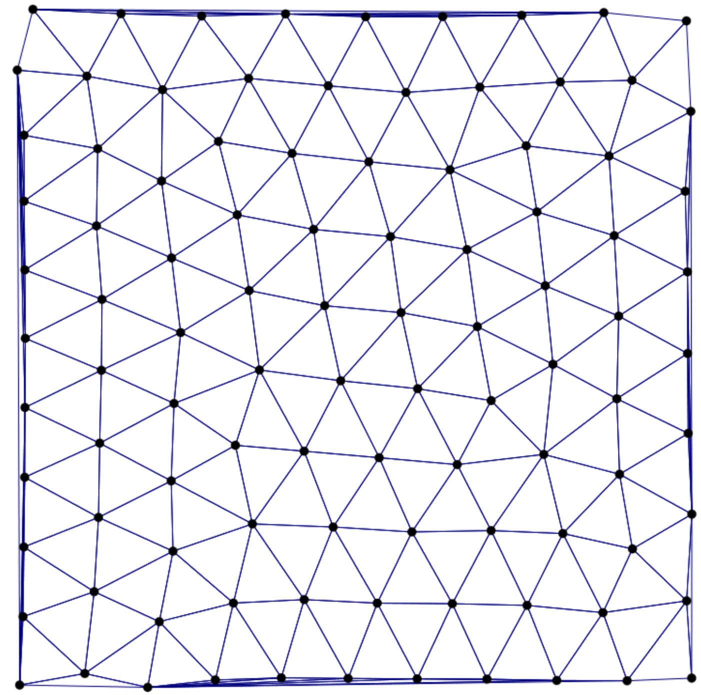
Back to the standard modeling pipeline…
Finite Element Modeling ?
How ?

Part. 3 From Geometry Processing to Applied Math.

Optimize a Voronoi diagram from the point of view of sampling regularity
(quantization noise power)



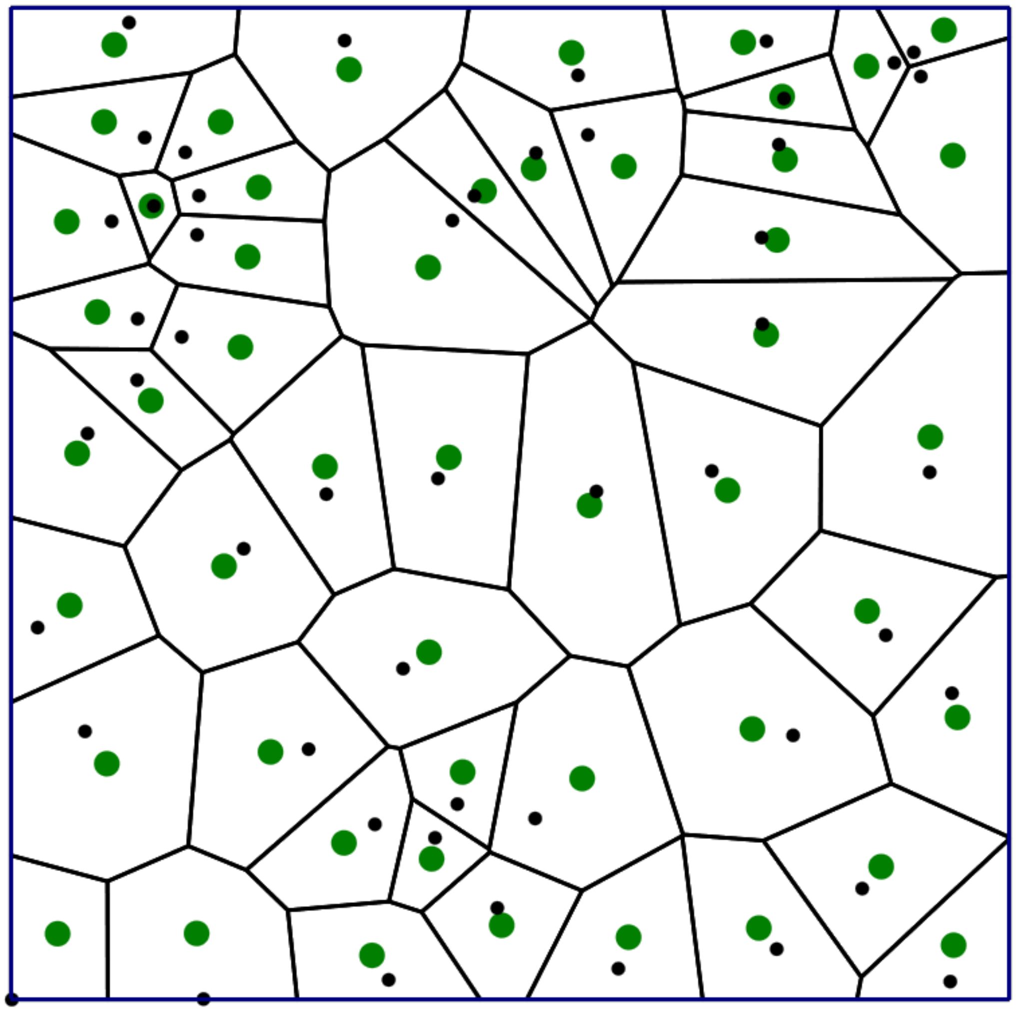
Minimize



$$F = \sum_i \int_{Vor(i)} \left\| \mathbf{x}_i - \mathbf{x} \right\|^2 d\mathbf{x}$$

Theorem: F is of class C^2 [Liu, Wang, **L**, Yan, Lu, ACM TOG 2008]

Part. 3 From Geometry Processing to Applied Math.



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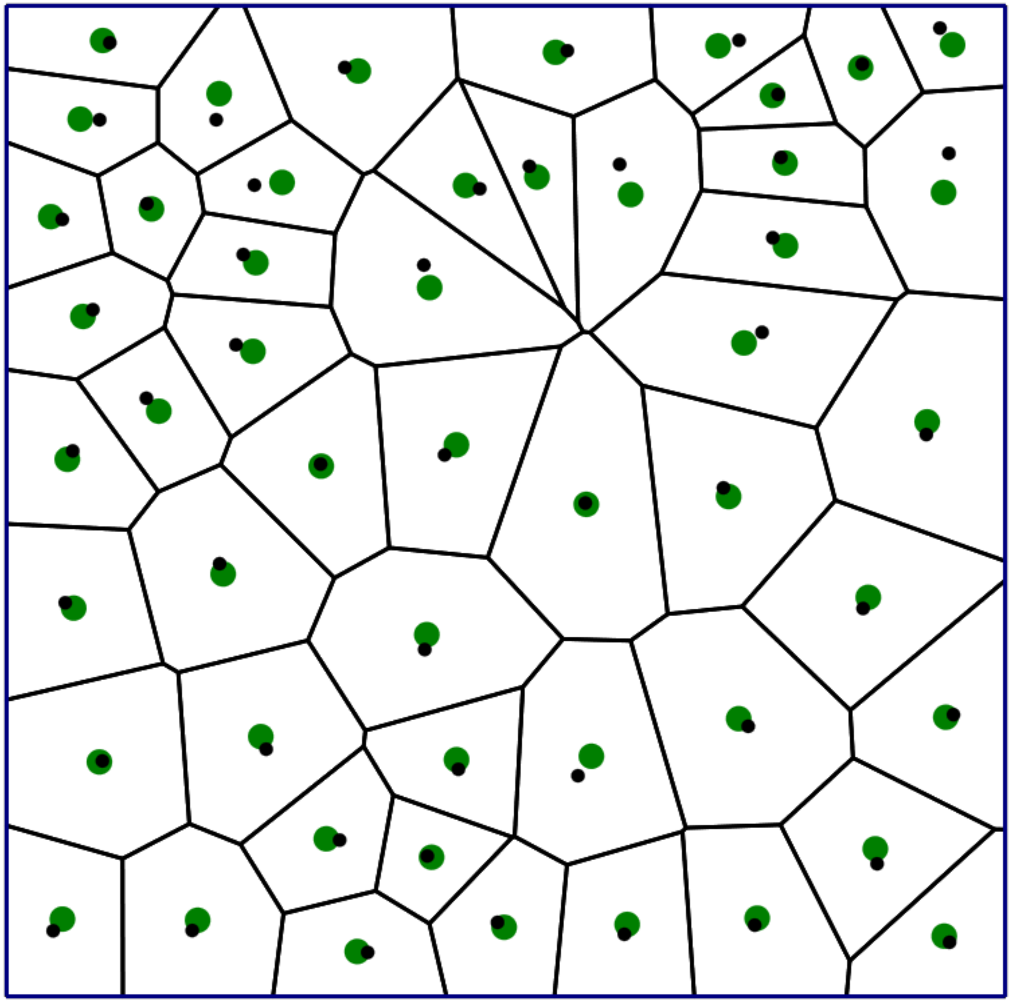
Part. 3 From Geometry Processing to Applied Math.



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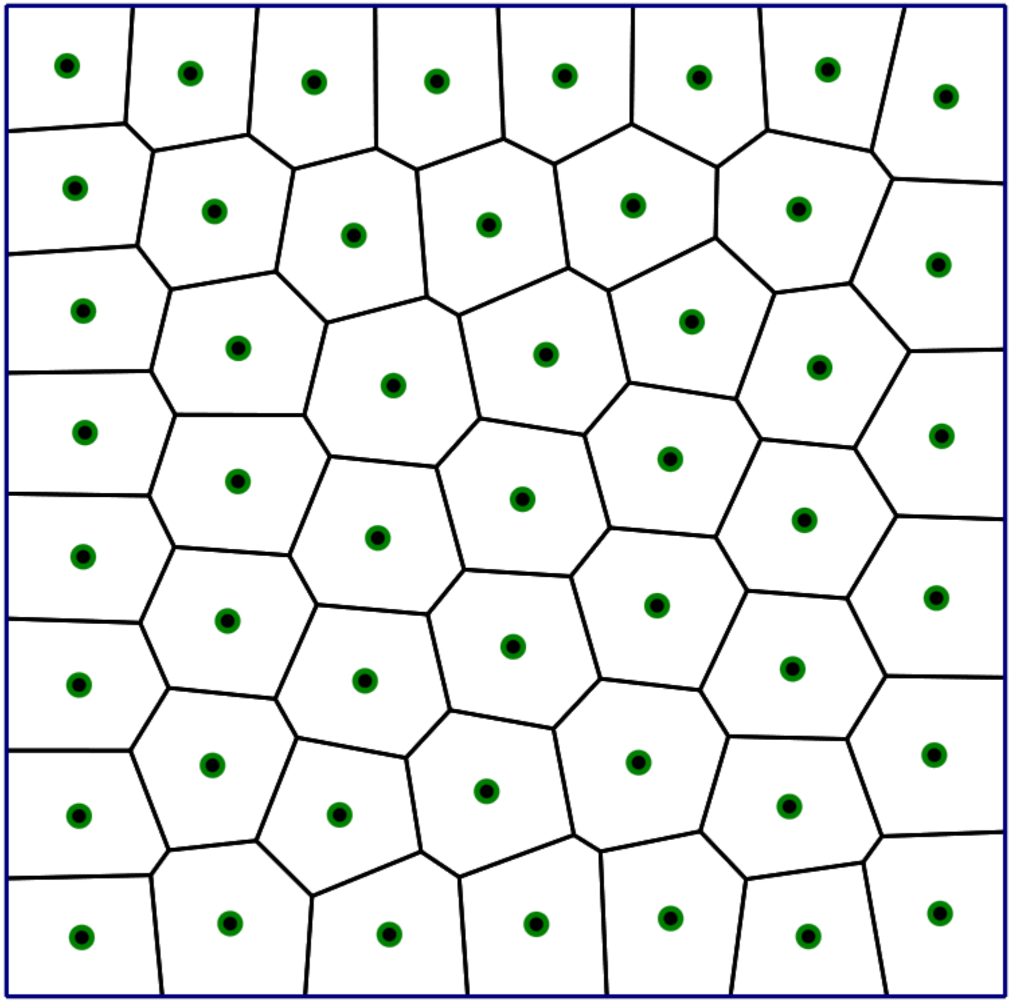
Part. 3 From Geometry Processing to Applied Math.



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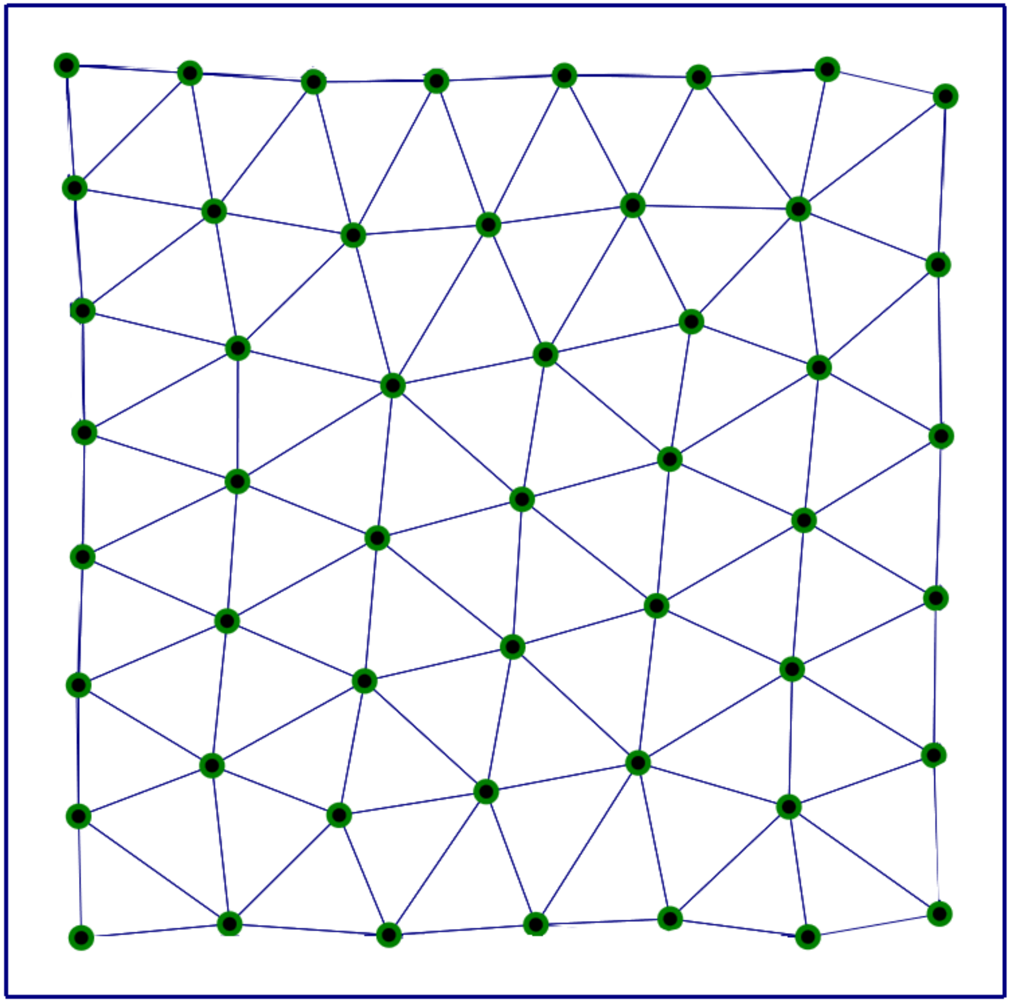
Part. 3 From Geometry Processing to Applied Math.



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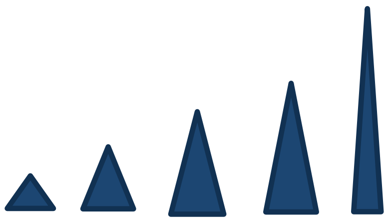
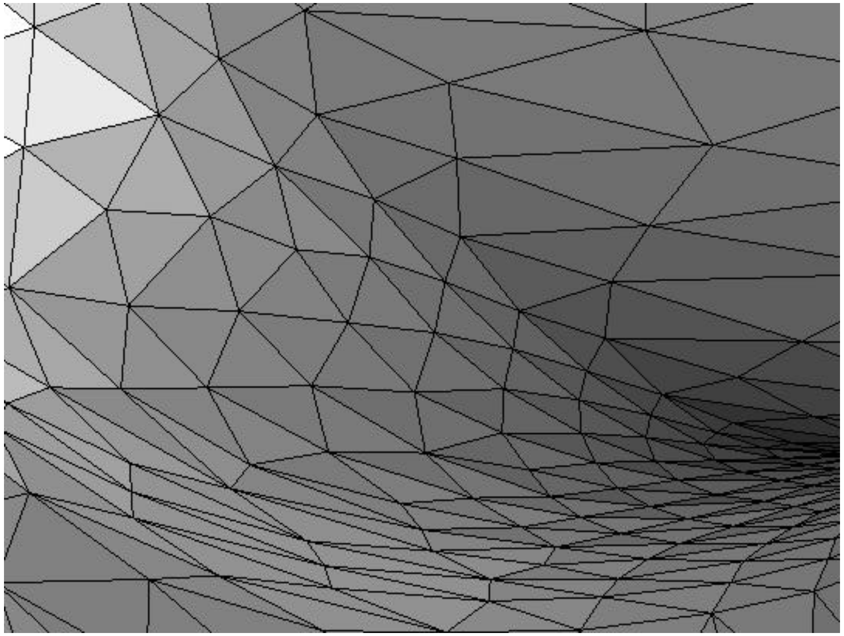
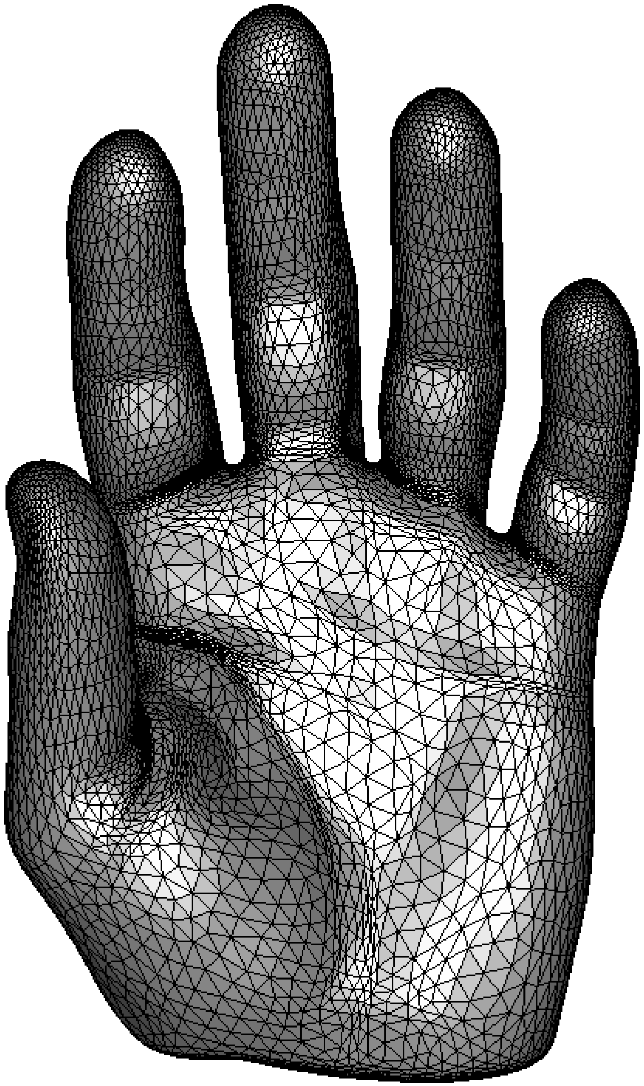
Part. 3 From Geometry Processing to Applied Math.

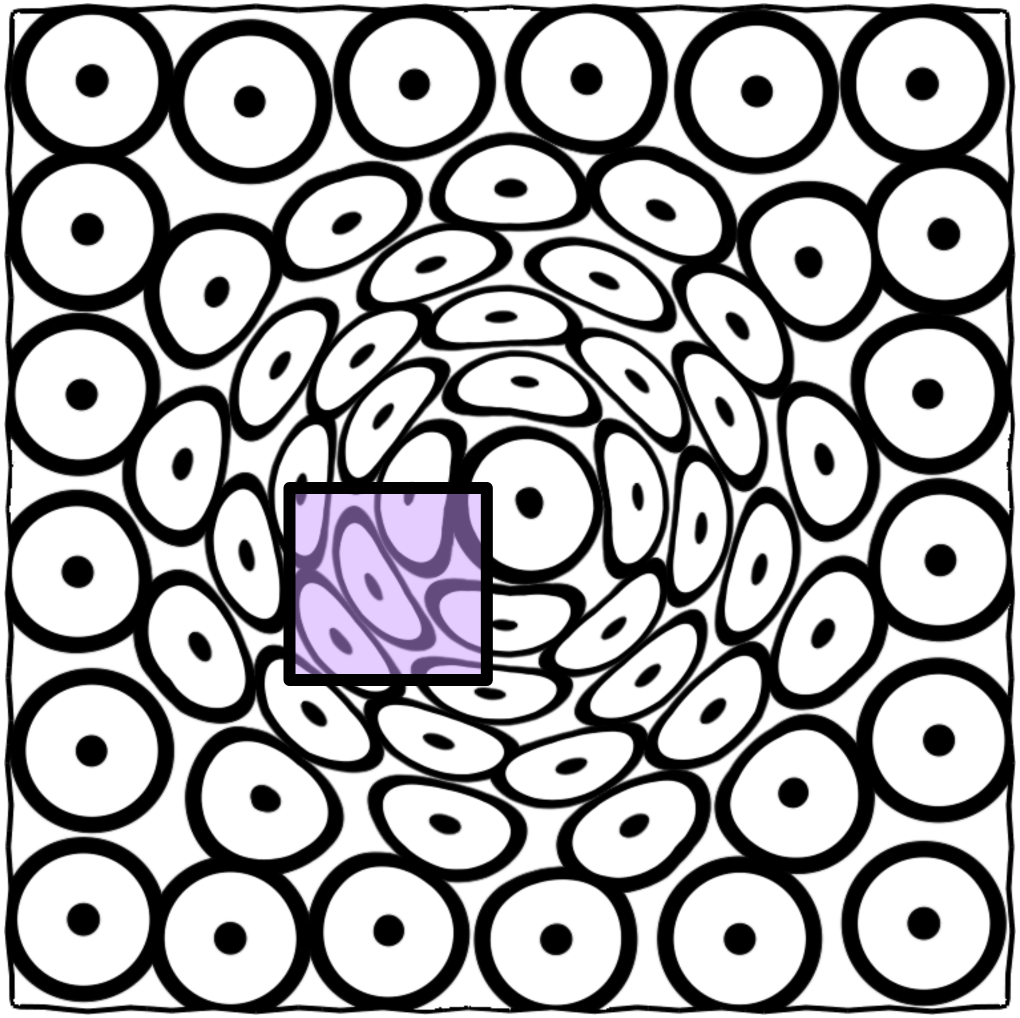


Anisotropic mesh:
* shape can vary
* size can vary



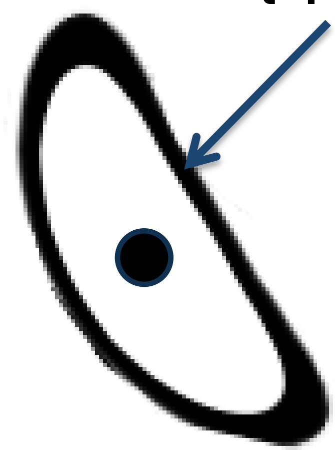


Part. 3. Anisotropy



The input: anisotropy field
Specifies shape and orientation

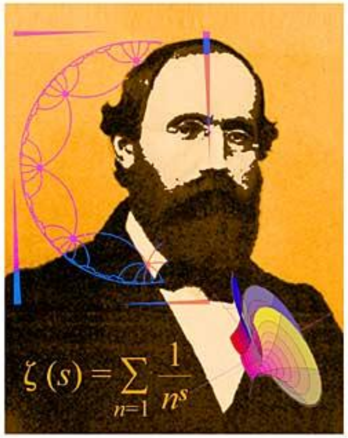
Anisotropy: An "alteration" of of distances and angles.



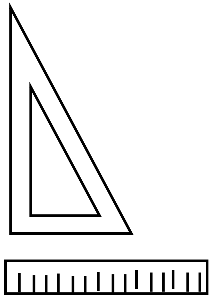
Part. 3. Anisotropy

The dot product: a geometric tool

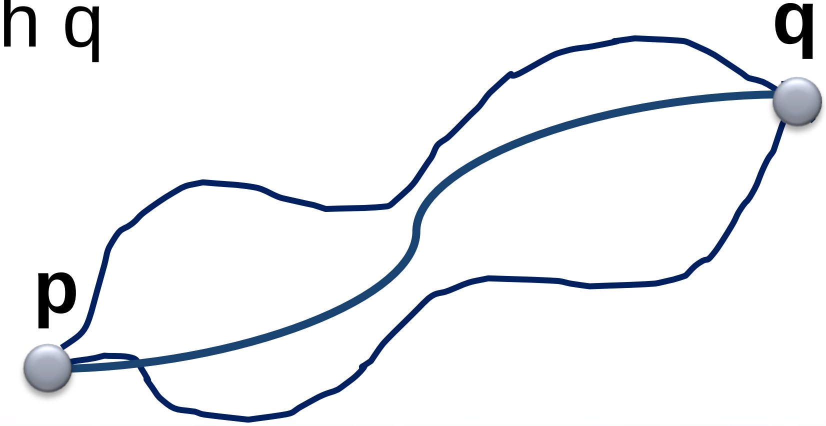
Anisotropic distance
between **p** *and* **q** *w.r.t.* G



Georg Friedrich Bernhard Riemann 1826 - 1866

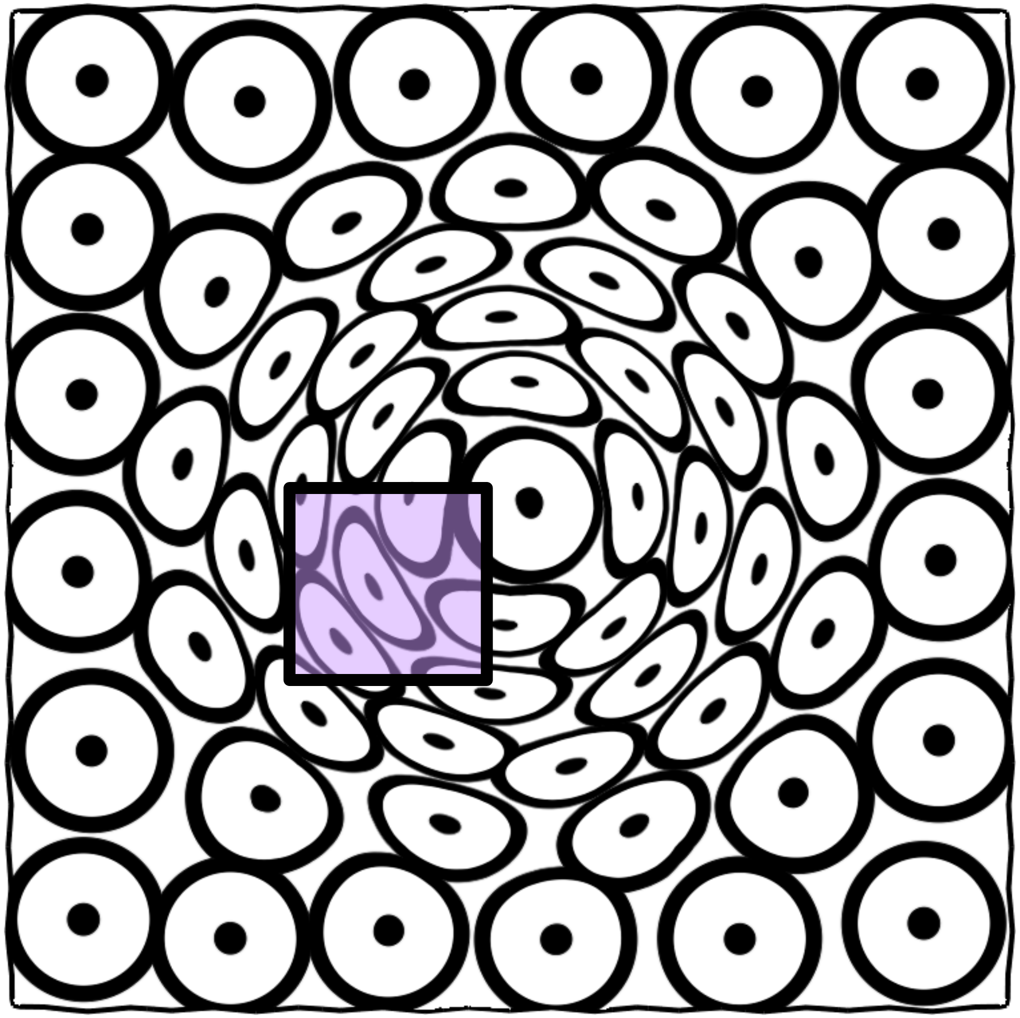


$d_G(\mathbf{p},\mathbf{q})$ = (anisotropic) length of
shortest curve
that connects p with q



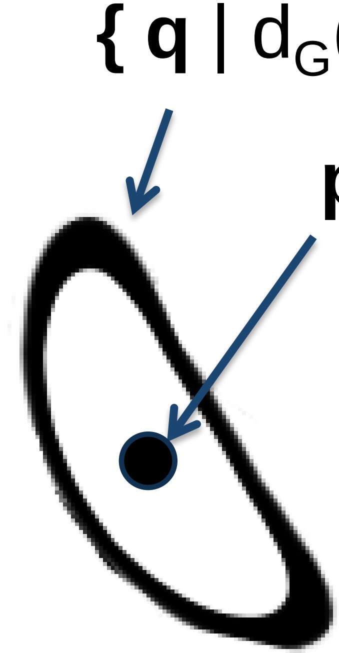
$$l_G(C) = \int_{t=0}^{1} \sqrt{v(t)^t \, G(t) \, v(t) \, dt}$$

Part. 3. Anisotropy

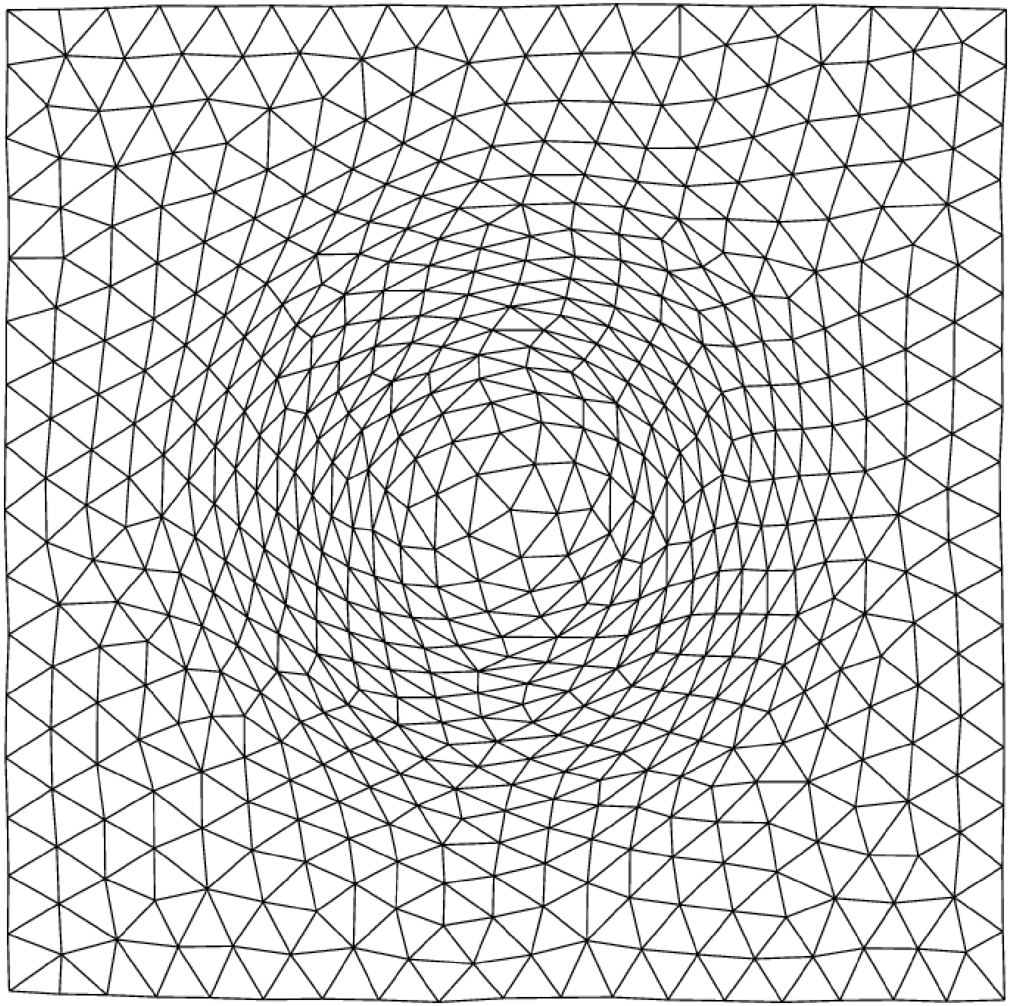


The input: anisotropy field

$$G(x,y) = \begin{bmatrix} a(x,y) & b(x,y) \\ b(x,y) & c(x,y) \end{bmatrix}$$

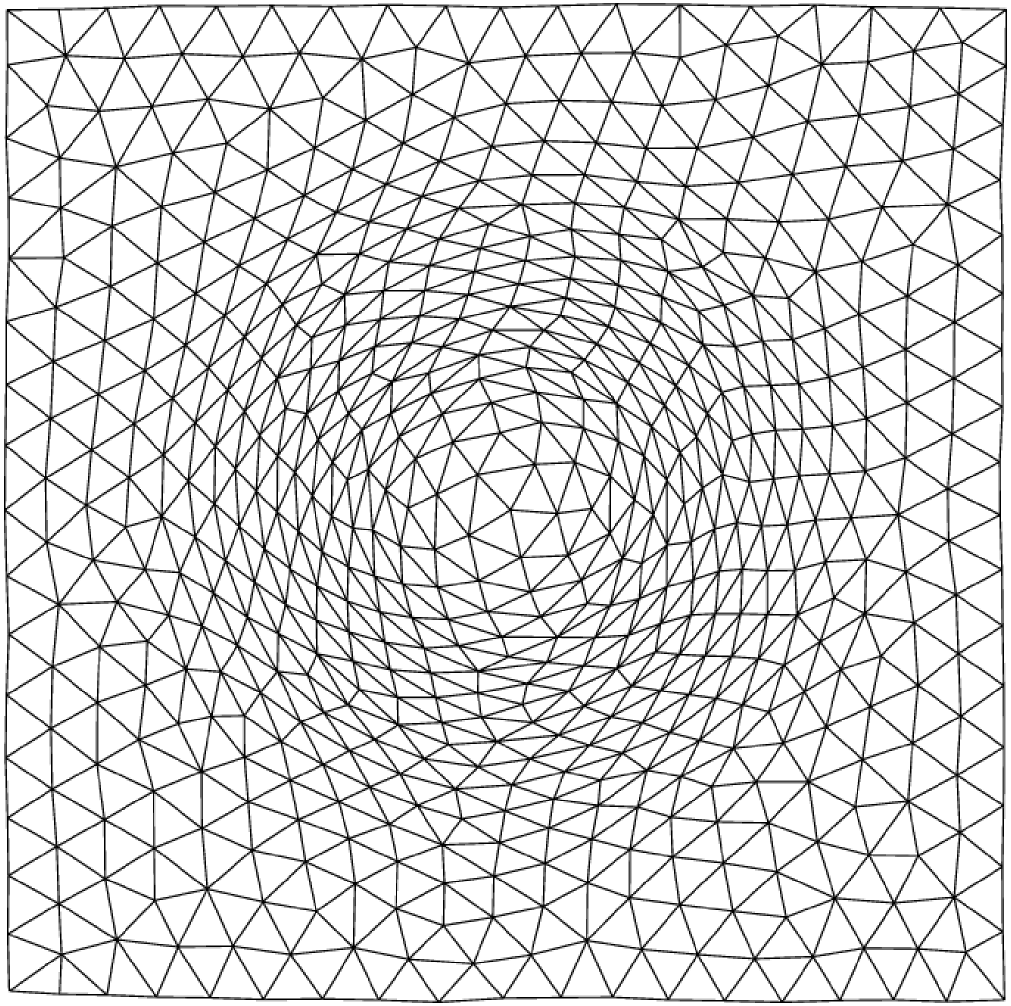


Part. 3. Anisotropy



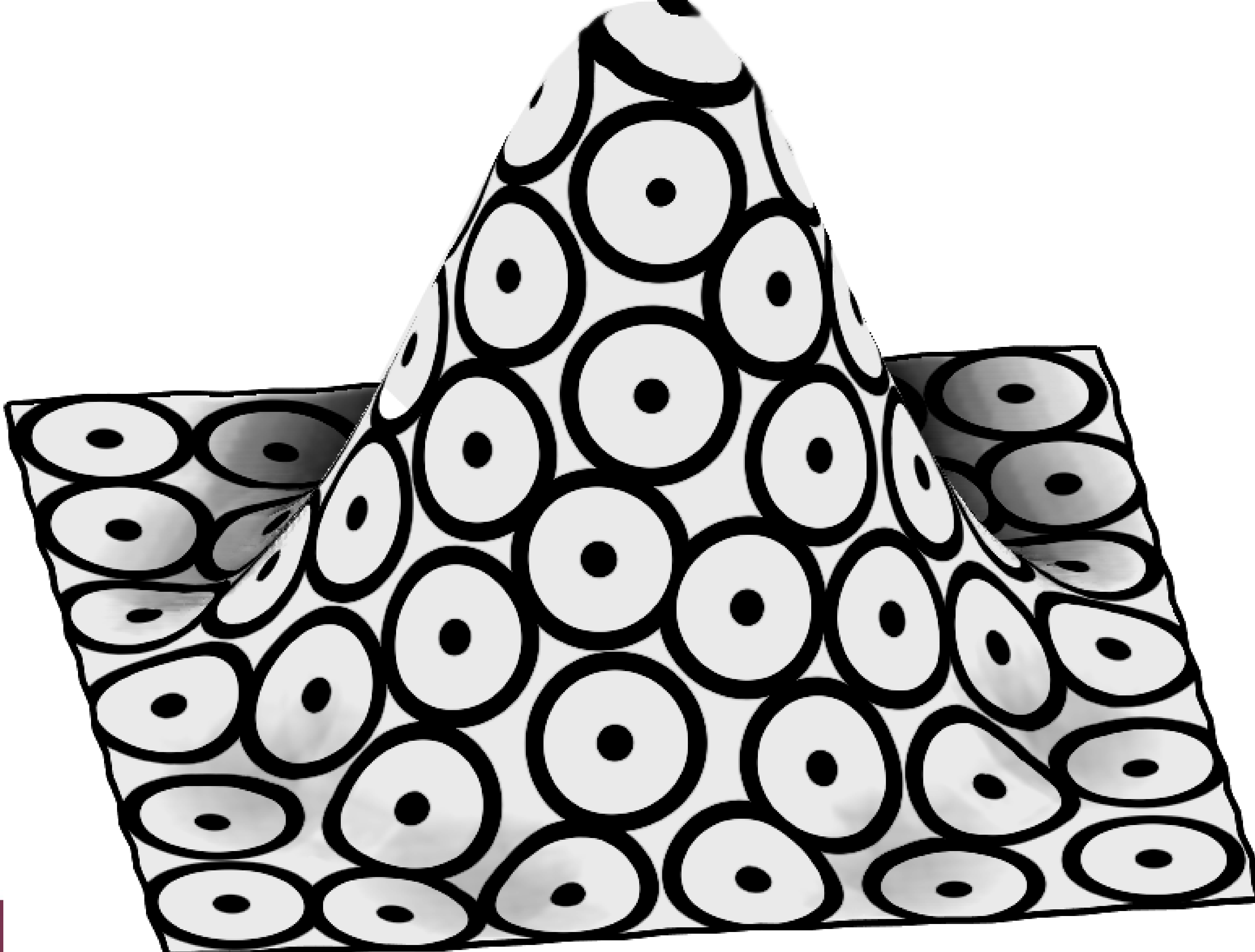
The result: triangles are "deformed" by the anisotropy.

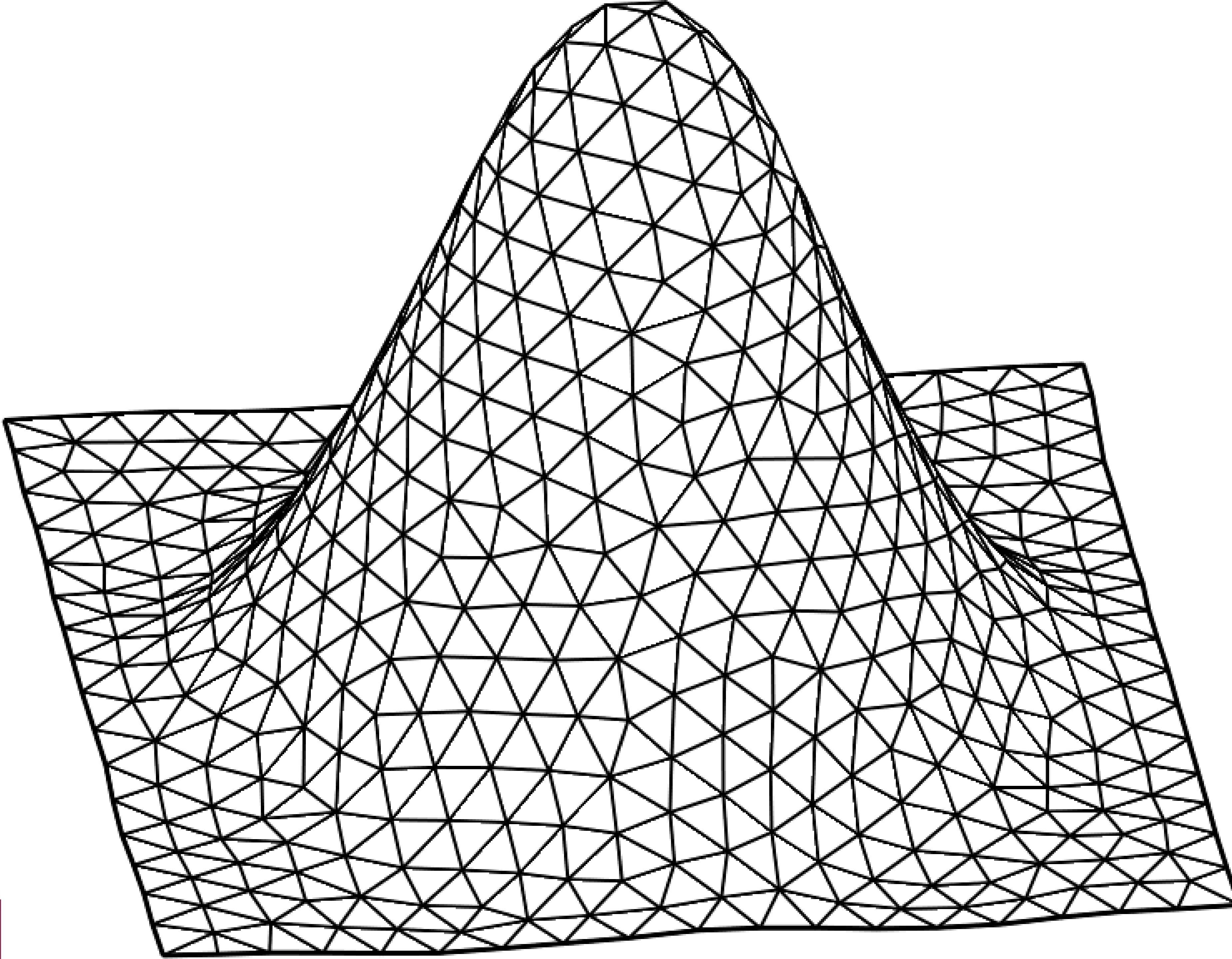
Part. 3. Anisotropy



The result: triangles are "deformed" by the anisotropy.

*Q: How to compute an **Anisotropic** Centroidal Voronoi Tessellation ?*





Part. 3 Journey in the 6^{th} dimension

The key idea

This example:

Anisotropic mesh in 2d ⟺ Isotropic mesh in 3d

Part. 3 Journey in the 6th dimension

The key idea

This example:

Anisotropic mesh in 2d $\Longleftrightarrow$ Isotropic mesh in 3d

Replace **anisotropy** with **additional dimensions**

Part. 3 Journey in the 6th dimension

The key idea

Replace **anisotropy** with **additional dimensions**

Note: more dimensions may be needed

Part. 3 Journey in the 6^{th} dimension

The key idea

Replace **anisotropy** with **additional dimensions**

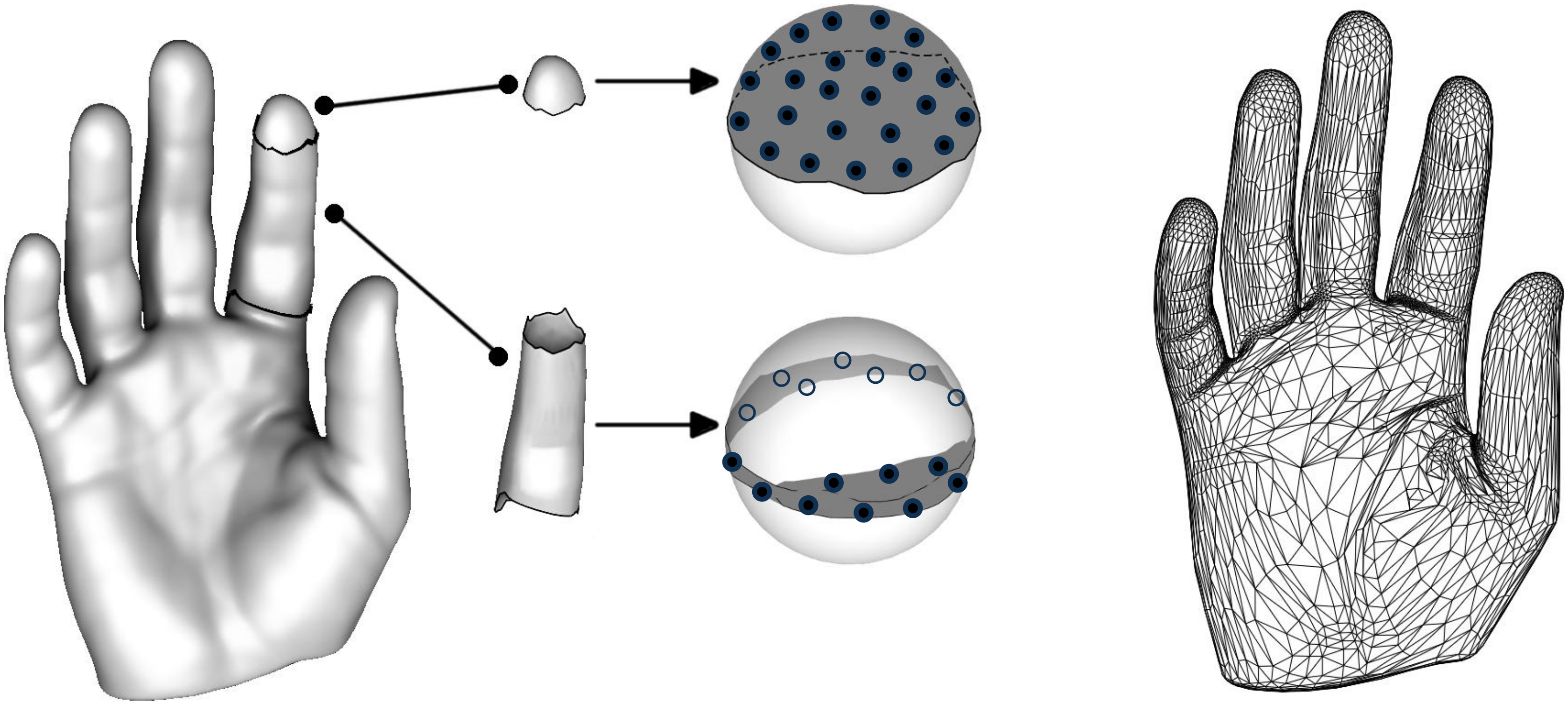
Note: more dimensions may be needed
How many ?
John Nash's isometric embedding theorem:

Maximum: depending on desired smoothness
C^1 : $2n$ [Nash-Kuiper]
C^k : bounded by $n(3n+11)/2$ [Nash, Nash-Moser]

Part. 3 Journey in the 6^{th} dimension

A 6d embedding for curvature-adapted meshing

The Gauss map: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \rightarrow \begin{bmatrix} N_x \\ N_y \\ N_z \end{bmatrix}$



Part. 3 Journey in the 6th dimension

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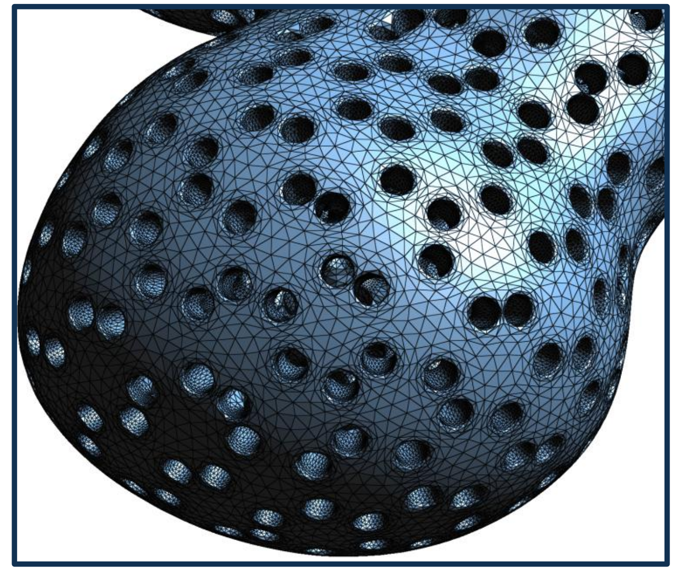
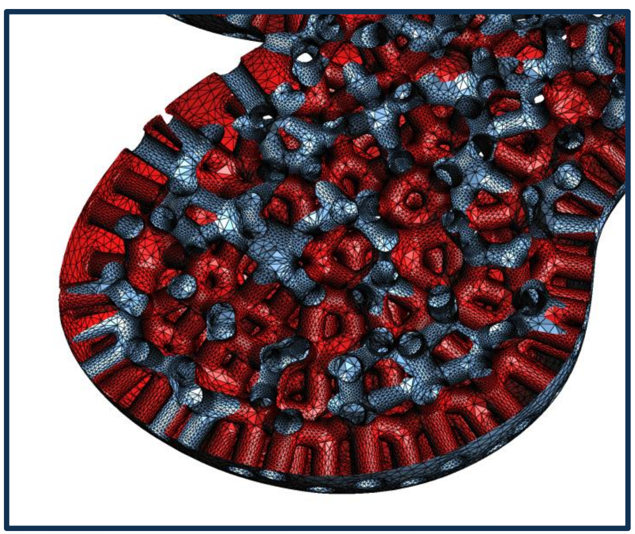
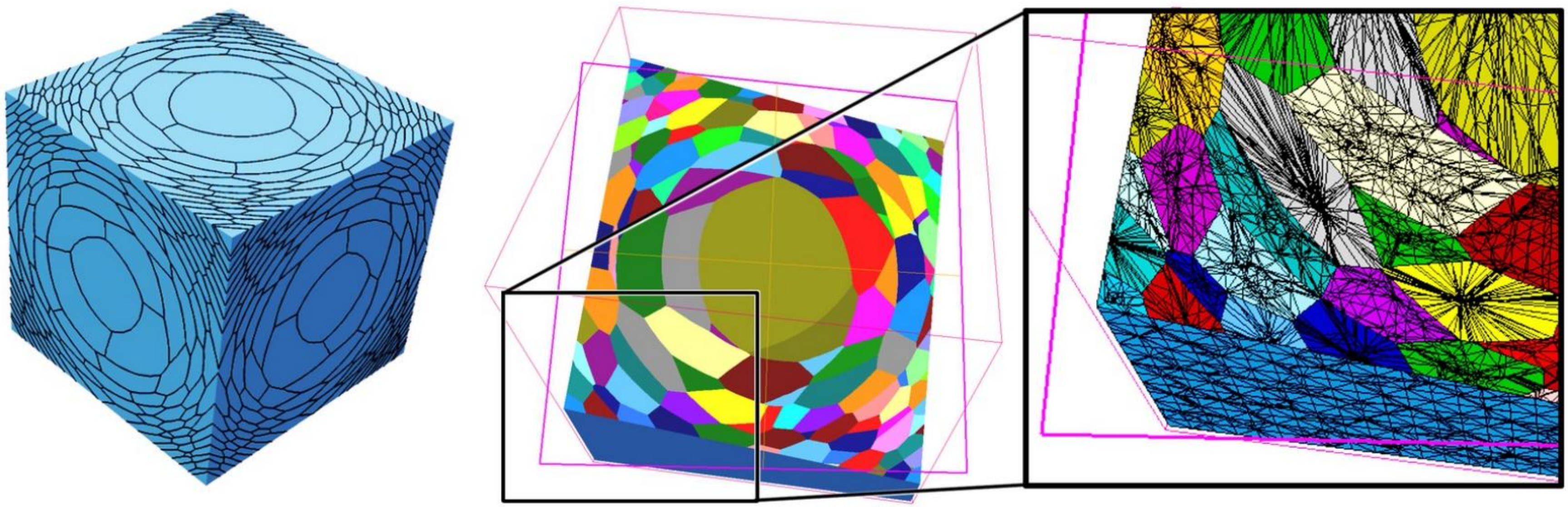




Vorpaline meshing software
ERC “Proof of Concept”

Part. 3 Journey in the 6th dimension

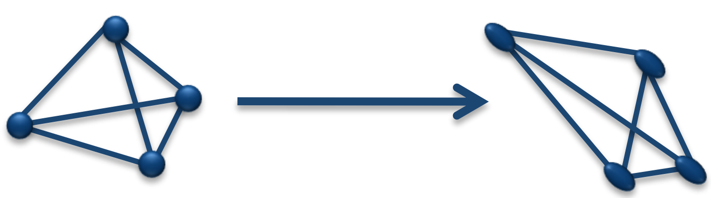
Anisotropy through high-dim. embedding



3D anisotropic Voronoi diagram and anisotropic Vector Quantization

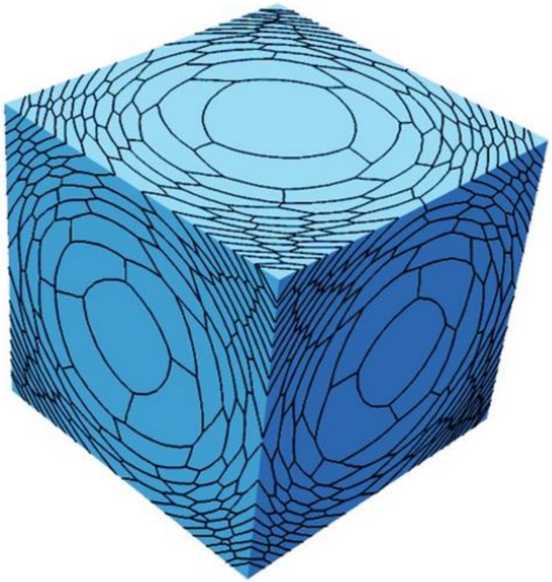
Anisotropy represented by a background mesh embedded in 6D

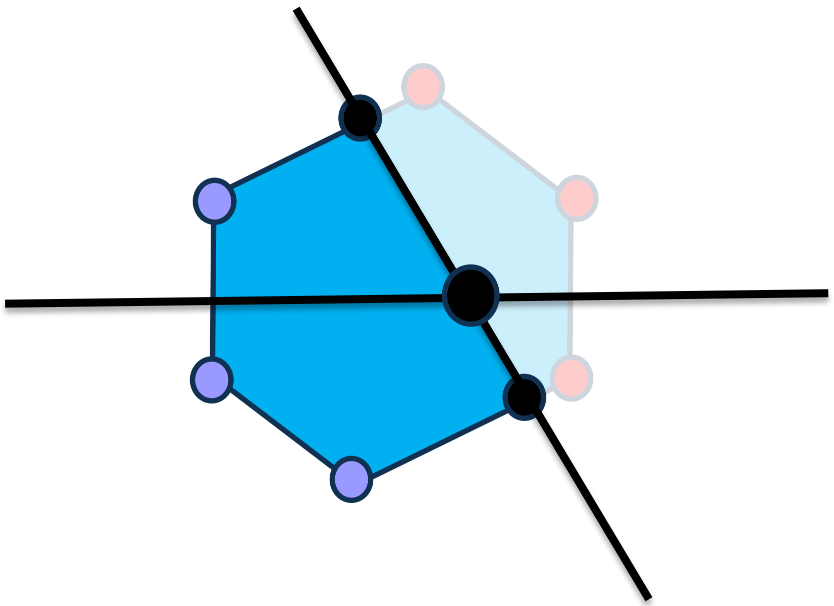
$$\mathbf{G_t = J_t^{\mathit{t}} J_t}$$



Part. 3 Journey in the 6th dimension

New predicates





$$
\begin{aligned}
side_1(\mathbf{p}_1,\mathbf{p}_2,\mathbf{q}) &= Sign(d^2(\mathbf{p}_2,\mathbf{q}) - d^2(\mathbf{p}_1,\mathbf{q}))\\
side_2(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{q}_1,\mathbf{q}_2) &= side_1(\mathbf{p}_1,\mathbf{p}_2,\mathbf{q}) \quad \text{where} \quad \mathbf{q} = \Pi(\mathbf{p}_1,\mathbf{p}_3)\cap[\mathbf{q}_1\mathbf{q}_2]\\
side_3(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4,\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3) &= side_1(\mathbf{p}_1,\mathbf{p}_2,\mathbf{q}) \quad \text{where} \quad \mathbf{q} = \Pi(\mathbf{p}_1,\mathbf{p}_3)\cap\Pi(\mathbf{p}_1,\mathbf{p}_4)\cap\Delta(\mathbf{q}_1\mathbf{q}_2\mathbf{q}_3)\\
side_4(\mathbf{p}_1,\ldots\mathbf{p}_5,\mathbf{q}_1,\ldots\mathbf{q}_4) &= side_1(\mathbf{p}_1,\mathbf{p}_2,\mathbf{q}) \ , \quad \mathbf{q} = \Pi(\mathbf{p}_1,\mathbf{p}_3)\cap\Pi(\mathbf{p}_1,\mathbf{p}_4)\cap\Pi(\mathbf{p}_1\mathbf{p}_5)\cap tet(\mathbf{q}_1\mathbf{q}_2\mathbf{q}_3\mathbf{q}_4)
\end{aligned}
$$

Part. 3 Journey in the 6th dimension

New predicates

```
Sign side2(
    point p0, point p1, point p2,
    point q0, point q1
) {

    scalar l1 = sq_dist(p1,p0) ;
    scalar l2 = sq_dist(p2,p0) ;

    scalar a10 = 2*dot_at(p1,q0,p0);
    scalar a11 = 2*dot_at(p1,q1,p0);
    scalar a20 = 2*dot_at(p2,q0,p0);
    scalar a21 = 2*dot_at(p2,q1,p0);

    scalar Delta = a11 - a10 ;
    scalar DeltaLambda0 = a11 - l1 ;
    scalar DeltaLambda1 =  l1 - a10 ;
    scalar r =
        Delta*l2-a20*DeltaLambda0-a21*DeltaLambda1 ;

    Sign Delta_sign = sign(Delta) ;
    Sign r_sign     = sign(r) ;

    generic_predicate_result(Delta_sign*r_sign) ;

    begin_sos3(p0,p1,p2)
       sos(p0, Sign(Delta_sign*sign(Delta-a21+a20)))
       sos(p1, Sign(Delta_sign*sign(a21-a20)))
       sos(p2, NEGATIVE)
    end_sos
}
```

Part. 3 Journey in the 6^{th} dimension

New predicates

```
Sign side2(
    point p0, point p1, point p2,
    point q0, point q1
) {

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    scalar a21 = 2*dot_at(p2,q1,p0);

    scalar Delta = a11 - a10 ;
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    scalar r =
        Delta*l2-a20*DeltaLambda0-a21*DeltaLambda1 ;

    Sign Delta_sign = sign(Delta) ;
    Sign r_sign     = sign(r) ;

    generic_predicate_result(Delta_sign*r_sign) ;

    begin_sos3(p0,p1,p2)
       sos(p0, Sign(Delta_sign*sign(Delta-a21+a20)))
       sos(p1, Sign(Delta_sign*sign(a21-a20)))
       sos(p2, NEGATIVE)
    end_sos
}
```
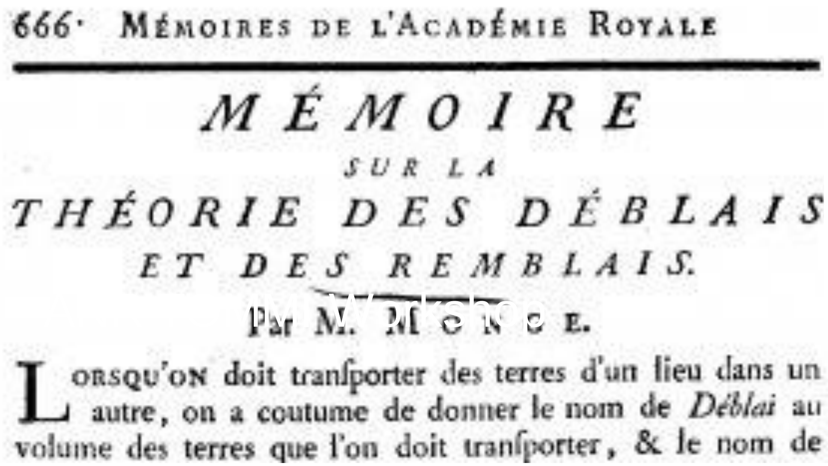

Automatically generated

```
Sign side2_exact_SOS(
    const double* p0,const double* p1,const double* p2,
    const double* q0,const double* q1,
    coord_index_t dim
) {
    const expansion& l1 = expansion_sq_dist(p1,p0,dim);
    const expansion& l2 = expansion_sq_dist(p2,p0,dim);
    const expansion& a10 = expansion_dot_at(p1,q0,p0,dim).scale_fast(2.0);
    const expansion& a11 = expansion_dot_at(p1,q1,p0,dim).scale_fast(2.0);
    const expansion& a20 = expansion_dot_at(p2,q0,p0,dim).scale_fast(2.0);
    const expansion& a21 = expansion_dot_at(p2,q1,p0,dim).scale_fast(2.0);
    const expansion& Delta = expansion_diff(a11,a10);
    Sign Delta_sign = Delta.sign(); vor_assert(Delta_sign != ZERO);

    const expansion& DeltaLambda0 = expansion_diff(a11,l1);
    const expansion& DeltaLambda1 = expansion_diff(l1,a10);
    const expansion& r0 = expansion_product(Delta,l2);
    const expansion& r1 = expansion_product(a20,DeltaLambda0).negate();
    const expansion& r2 = expansion_product(a21,DeltaLambda1).negate();
    const expansion& r = expansion_sum3(r0,r1,r2);
    Sign r_sign = r.sign();

    if(r_sign == ZERO) {
        const double* p_sort[3];
        p_sort[0] = p0;
        p_sort[1] = p1;
        p_sort[2] = p2;
        std::sort(p_sort,p_sort + 3);
        for(index_t i = 0; i < 3; ++i) {
            if(p_sort[i] == p0) {
                const expansion& z1 = expansion_diff(Delta,a21);
                const expansion& z = expansion_sum(z1,a20);
                Sign z_sign = z.sign();
                if(z_sign != ZERO) { return Sign(Delta_sign * z_sign); }
            }
            if(p_sort[i] == p1) {
                const expansion& z = expansion_diff(a21,a20);
                Sign z_sign = z.sign();
                if(z_sign != ZERO) { return Sign(Delta_sign * z_sign); }
            }
            if(p_sort[i] == p2) {
                return NEGATIVE;
            }
        }
        vor_assert_not_reached;
    }
    return Sign(Delta_sign * r_sign);
}
```

Predicate Construction Toolkit [PCK] – make it easy for everybody

Part. 3 Optimal Transport

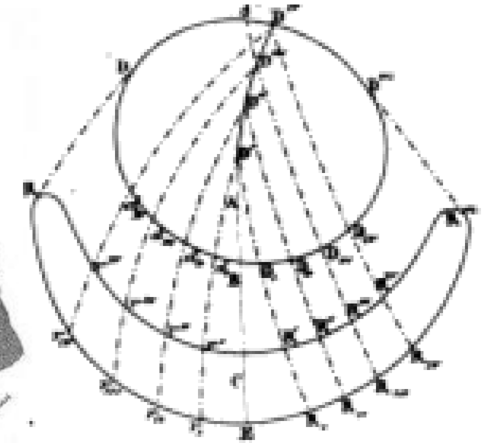
Gaspard Monge - 1784

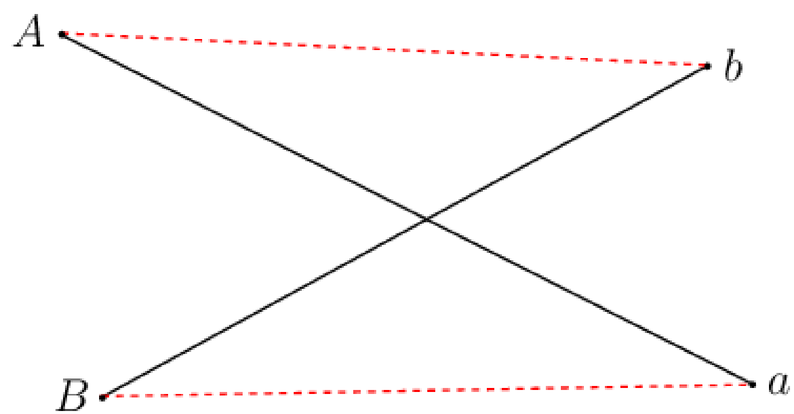

666. MÉMOIRES DE L'ACADÉMIE ROYALE

MÉMOIRE
SUR LA
THÉORIE DES DÉBLAIS
ET DES REMBLAIS.

Par M. MONGE.

Lorsqu'on doit tranſporter des terres d'un lieu dans un autre, on a coutume de donner le nom de *Déblai* au volume des terres que l'on doit tranſporter, & le nom de





Part. 3 Optimal Transport – some references

A Multiscale Approach to Optimal Transport,
Quentin Mérigot, Computer Graphics Forum, 2011

Variational Principles for Minkowski Type Problems, Discrete Optimal Transport, and Discrete Monge-Ampere Equations
Xianfeng Gu, Feng Luo, Jian Sun, S.-T. Yau, ArXiv 2013

Minkowski-type theorems and least-squares clustering
AHA! (Aurenhammer, Hoffmann, and Aronov), SIAM J. on math. ana. 1998

Topics on Optimal Transportation, 2003
Optimal Transport Old and New, 2008
Cédric Villani

Yann Brénier, Jean-David Benamou

Part. 3 Optimal Transport – Monge's problem

μ

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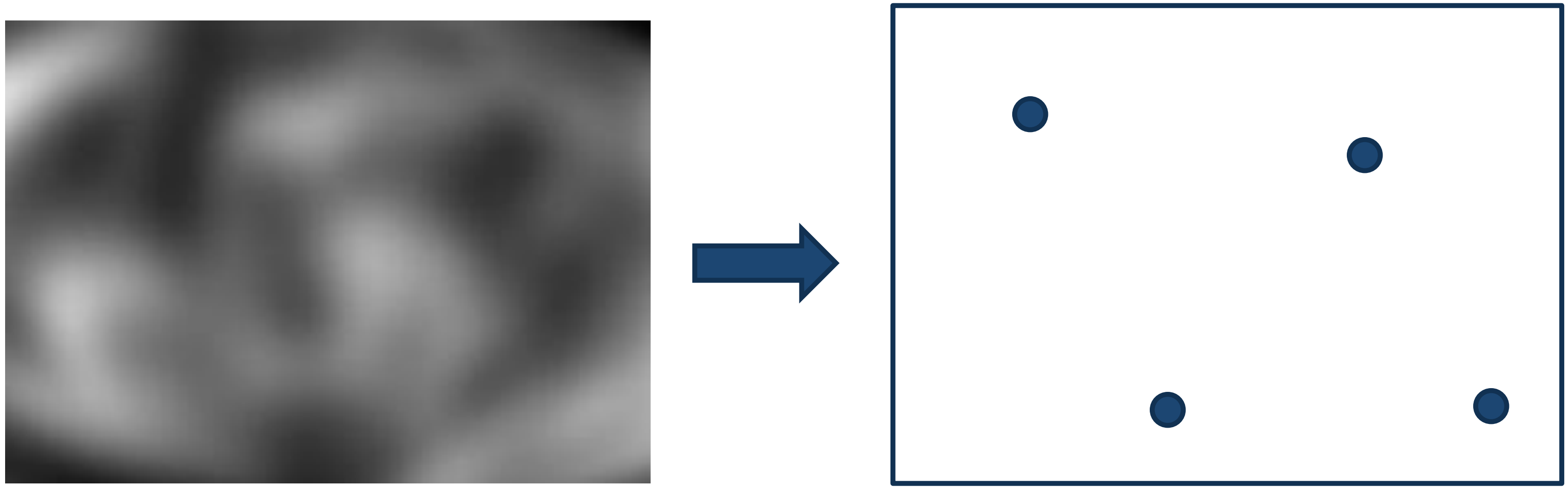
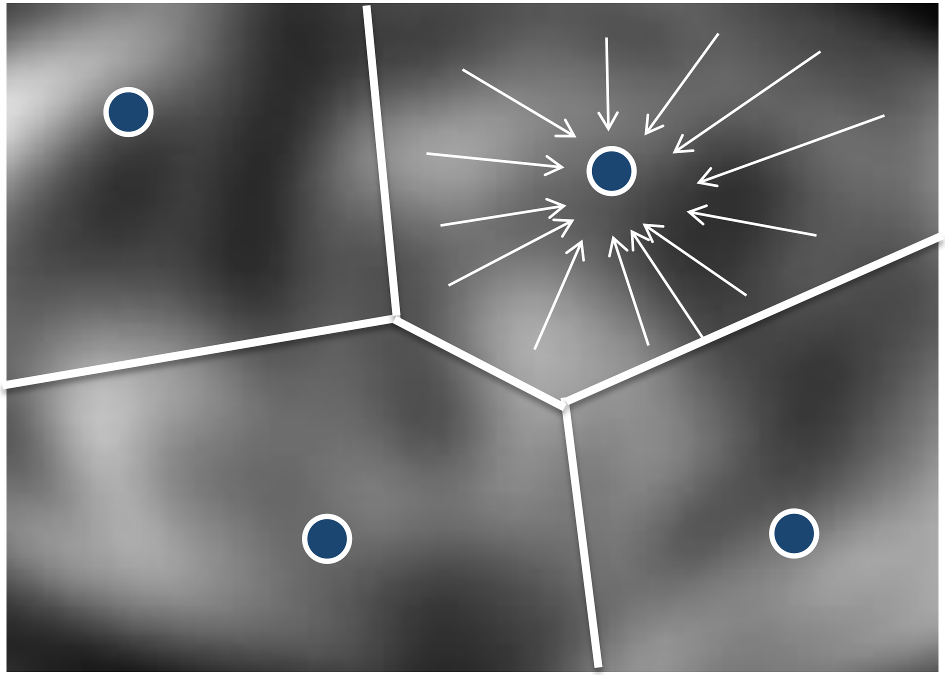
Monge's problem:

Find a transport map T that minimizes $C(T) = \int_{\Omega} || x - T(x) ||^2 \, d\mu(x)$

Part. 3 Optimal Transport – semi-discrete

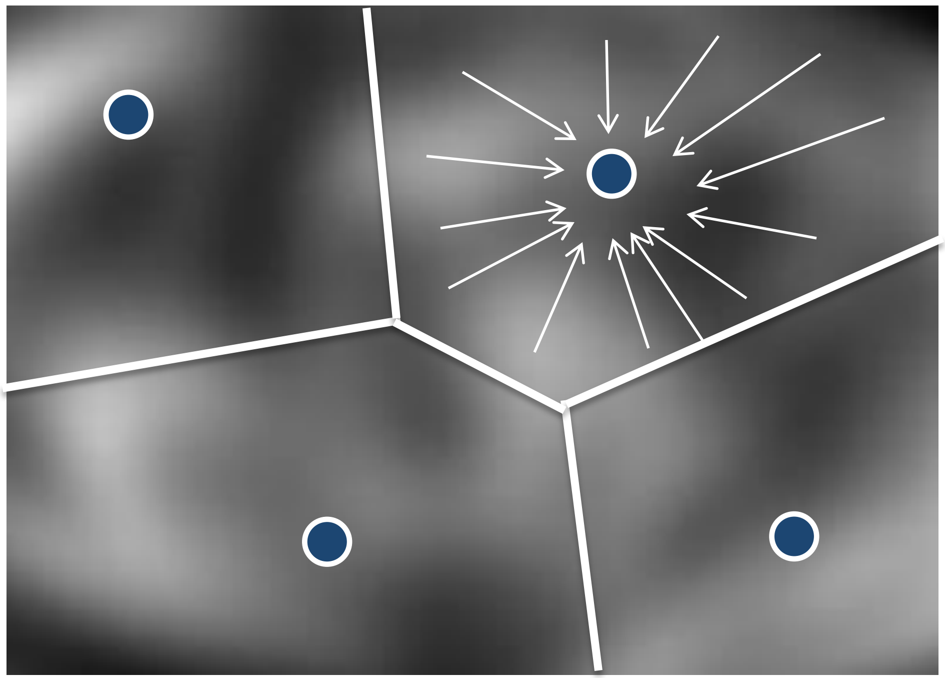


Part. 3 Optimal Transport – semi-discrete



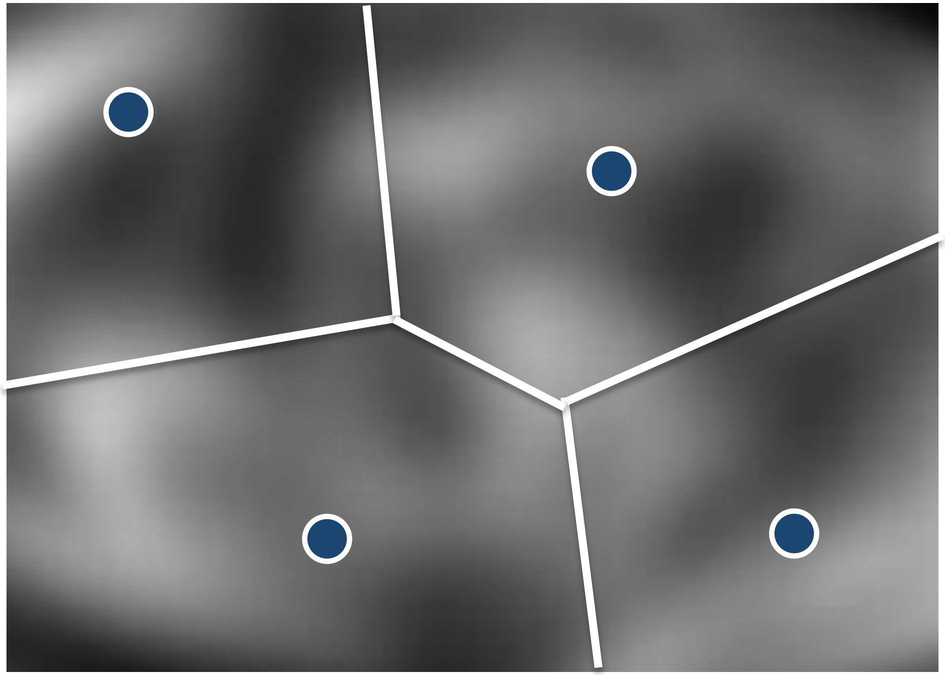
The pre-images of the Diracs define a partition of Ω

Part. 3 Optimal Transport – semi-discrete



The pre-images of the Diracs define a partition of Ω
This partition is a **power diagram** (more on this below)

Part. 3 Optimal Transport – the AHA paper



Theorem **[Aurenhammer, Hoffmann, Aronov 98], [Brenier91]**:

given a measure μ with density, a set of points (S), a set of positive coefficients λ such that $\sum \lambda_i = \int d\mu(x)$, it is possible to find the weights w such that the map $T_S{}^W$ is an optimal transport map between μ and $\nu = \sum \lambda_i \, \delta(s_i)$

Given the points (S), one can find the weights (w) such that $\int_{Pow(si)} d\mu(x) = \lambda_i$

Part. 3 Optimal Transport – the algorithm

The [AHA] paper summary:

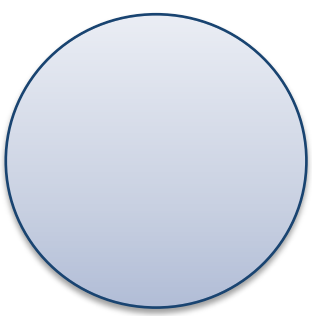
- The optimal weights minimize a convex function
- The gradient of this convex function is easy to compute

Note: the weight w(s) correspond to the Kantorovich potential $\psi(x)$
(solves a “discrete Monge-Ampere” equation)

The algorithm:

Summary:

The algorithm computes the weights w_i such that the power cells associated with the Diracs correspond to the preimages of the Diracs.



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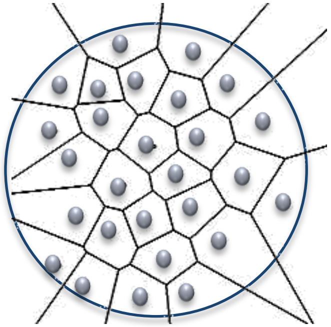

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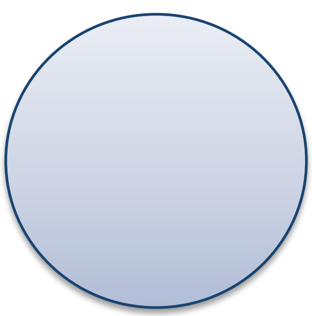

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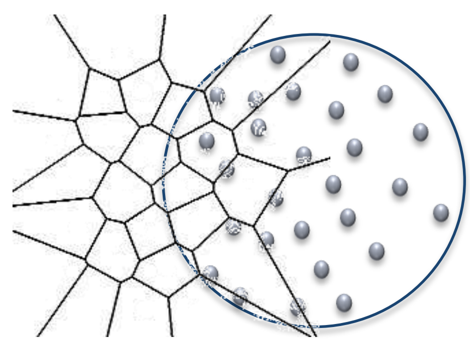

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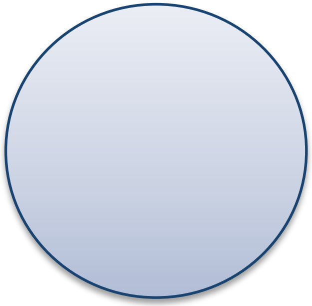

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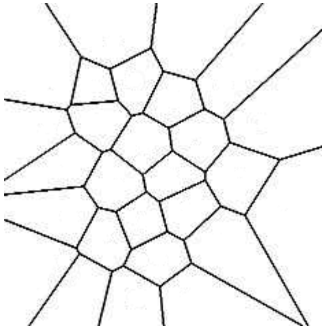

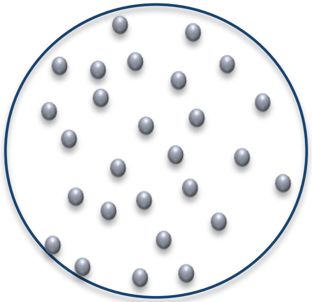

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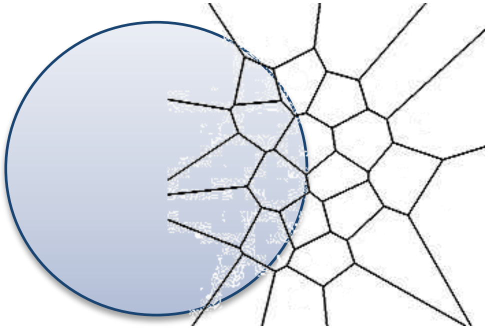

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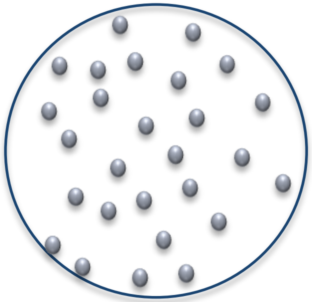

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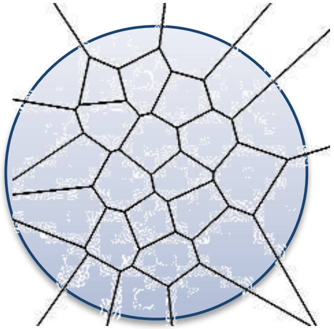

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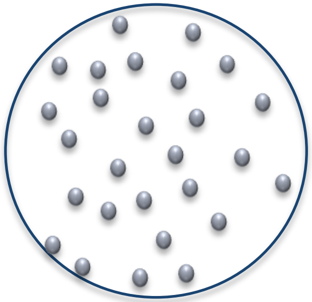

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<u>Wait a minute:</u>

This means that one can move (and possibly deform) a power diagram simply by changing the weights ?

Wait a minute:

This means that one can move (and possibly deform) a power diagram simply by changing the weights ?

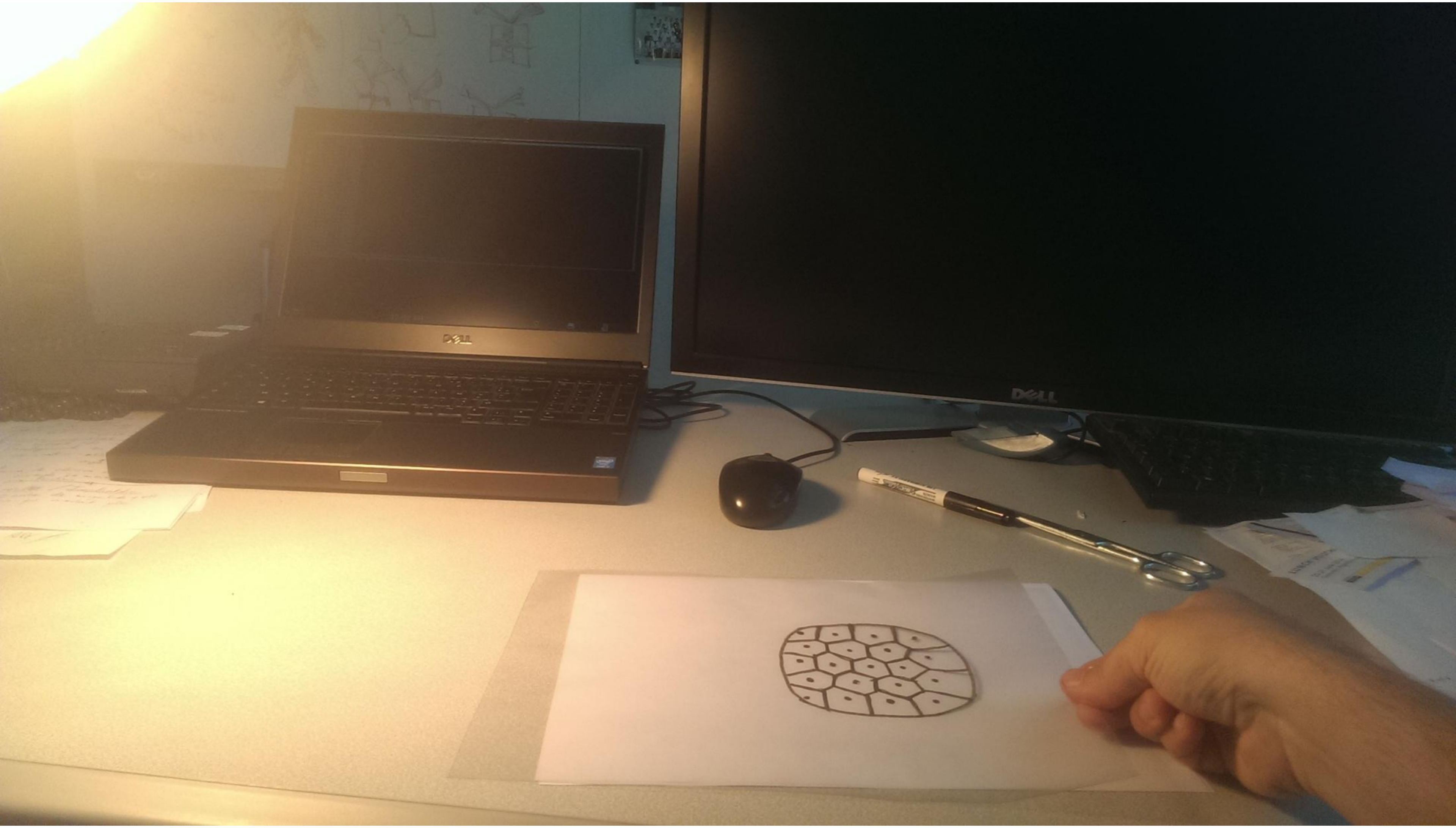
Reminder: Power diagram in 2d = intersection between Voronoi diagram in 3d and $\mathbb{R}^2$

$h_i = \mathrm{sqrt}(w_{max} - w_i)$

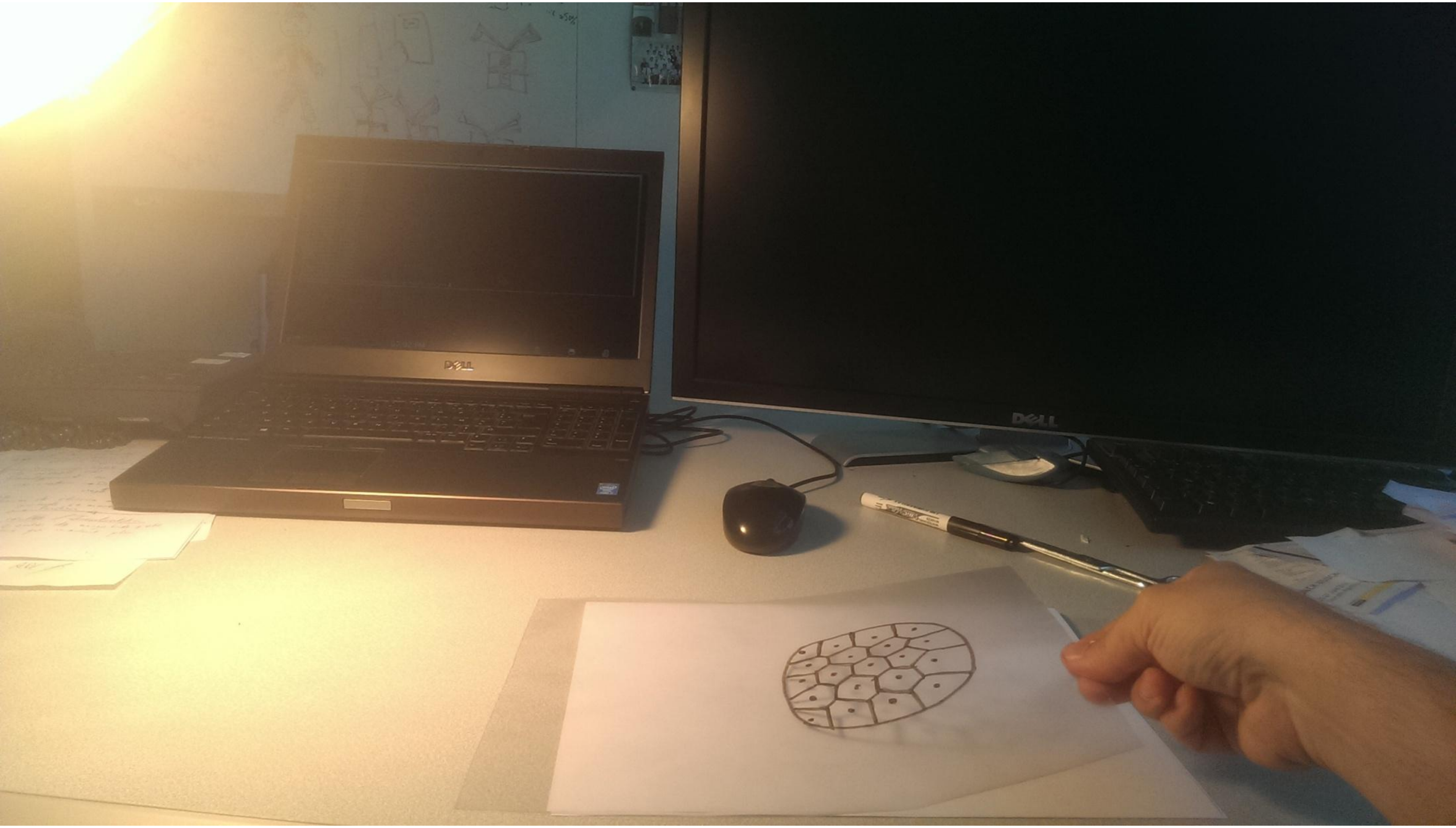
Height of point i

Weight of point i

Part. 3 Power Diagrams & Transport



Part. 3 Power Diagrams & Transport



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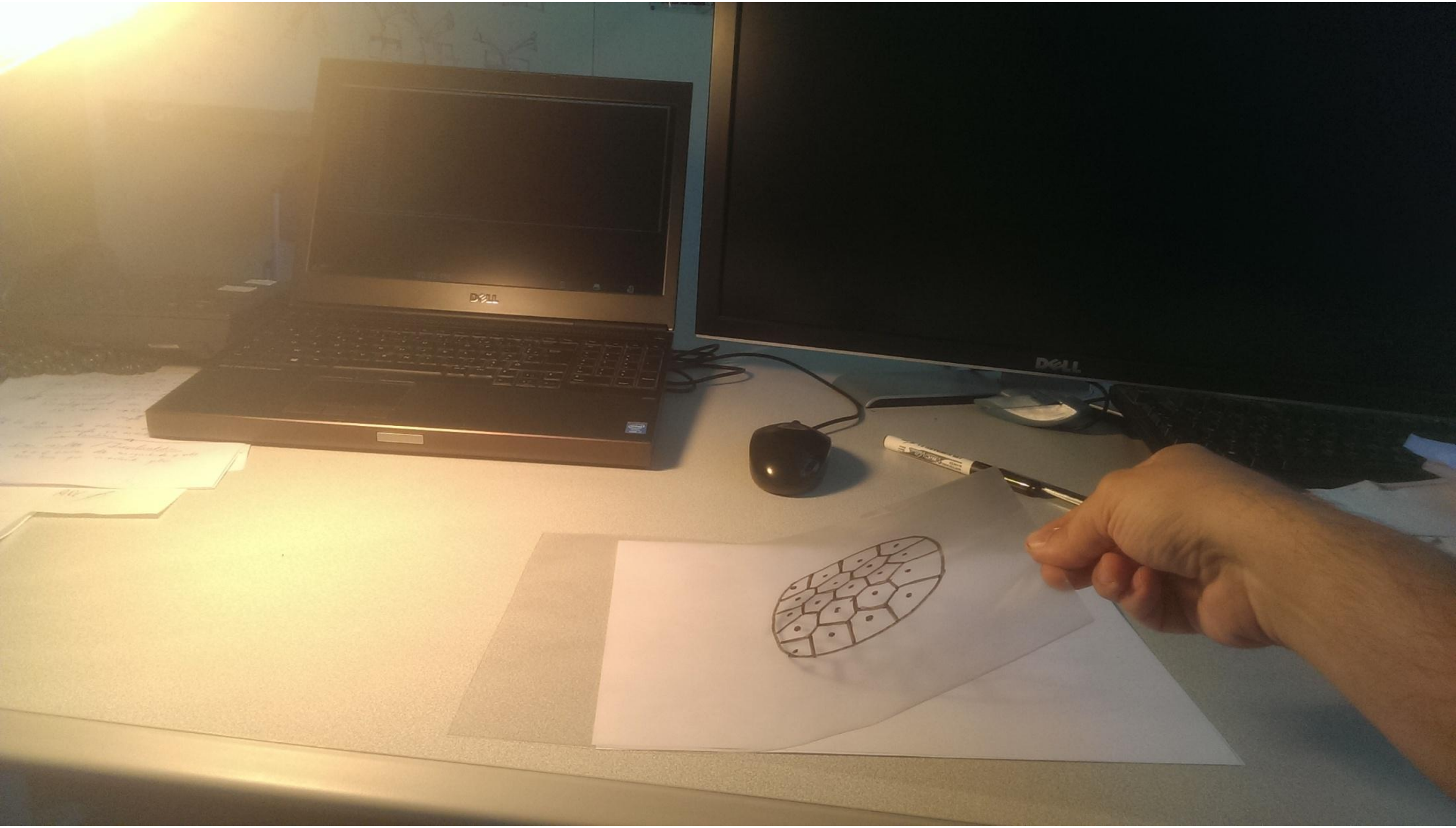
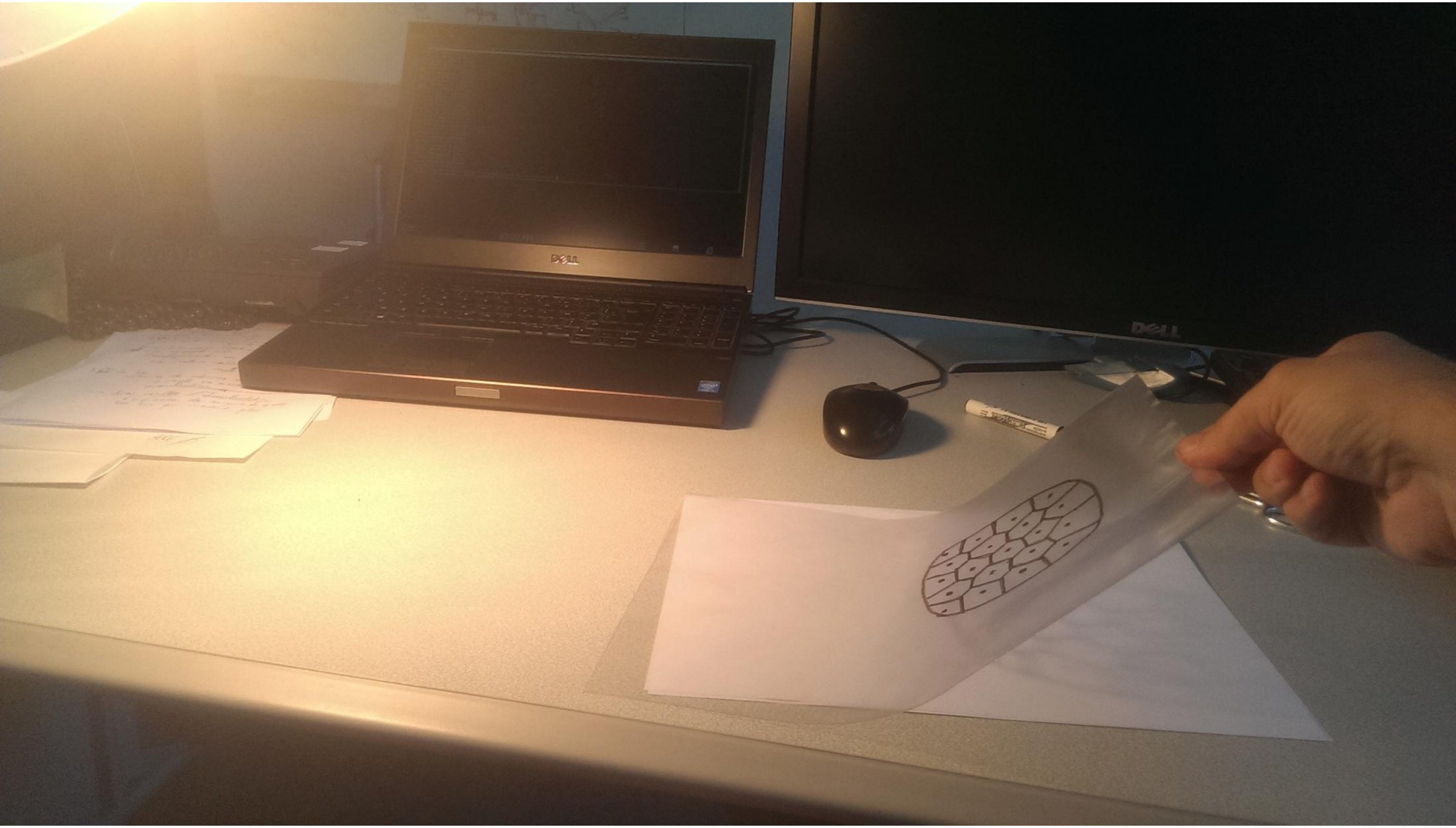
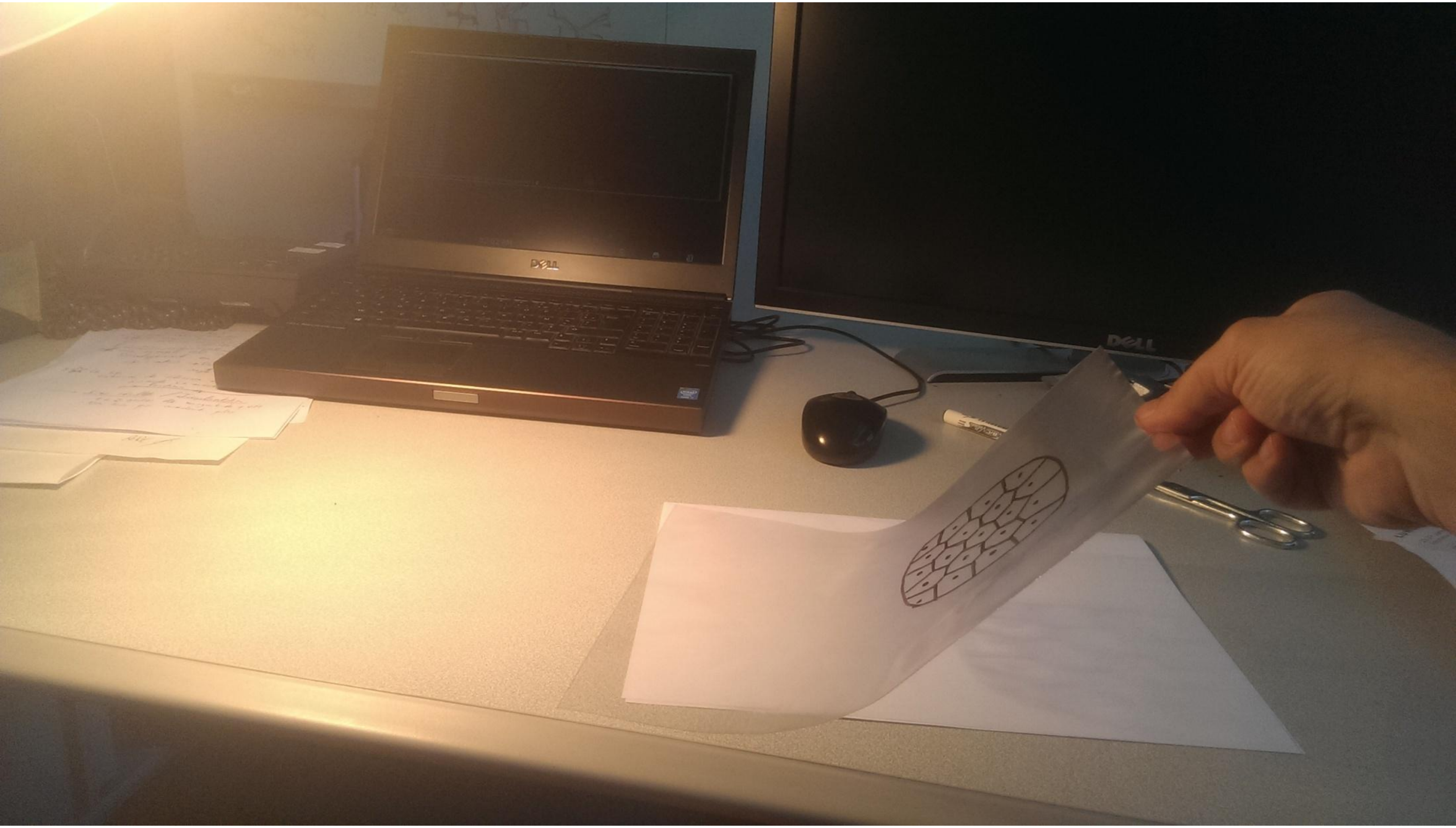


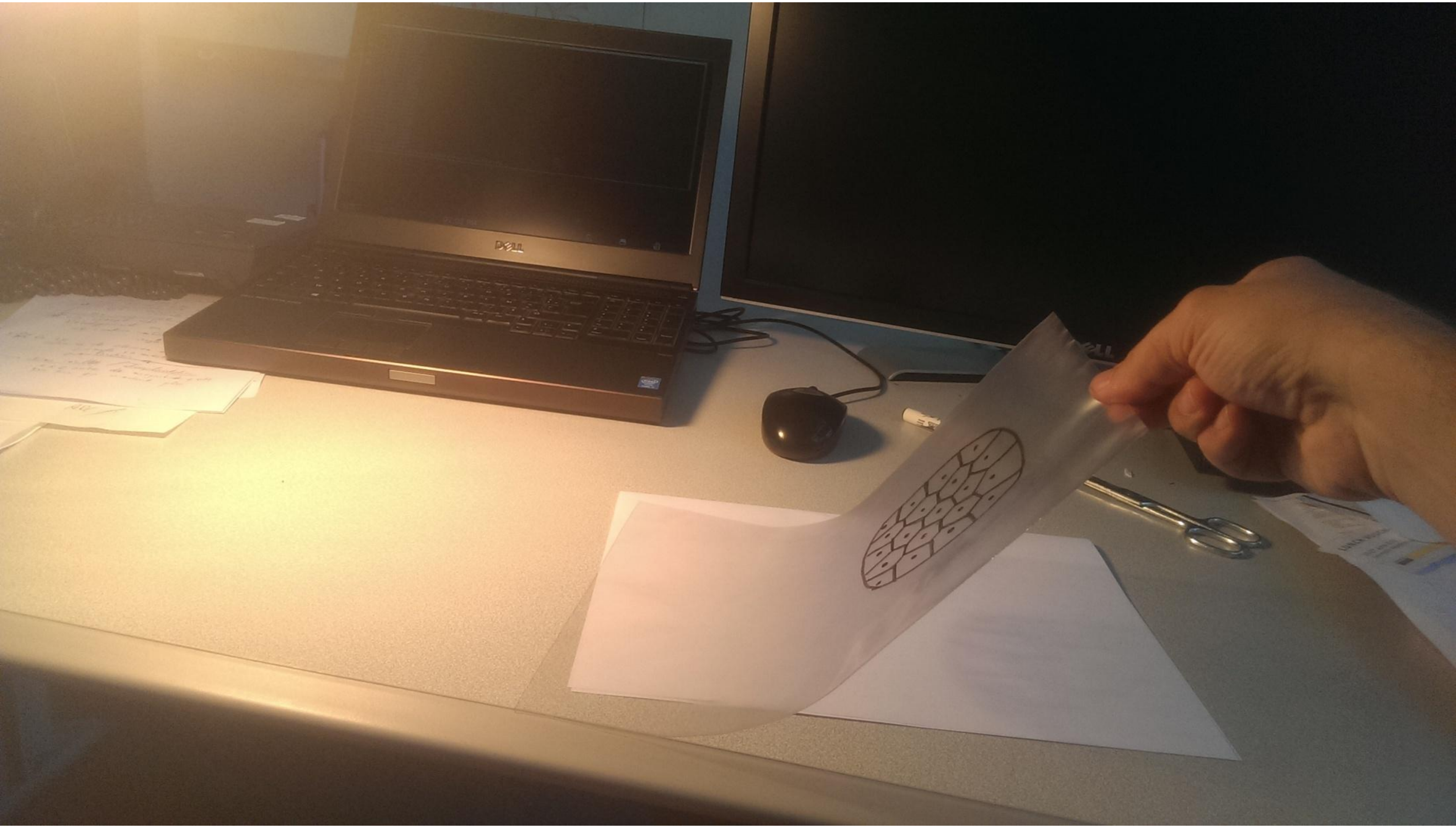
Part. 3 Power Diagrams & Transport



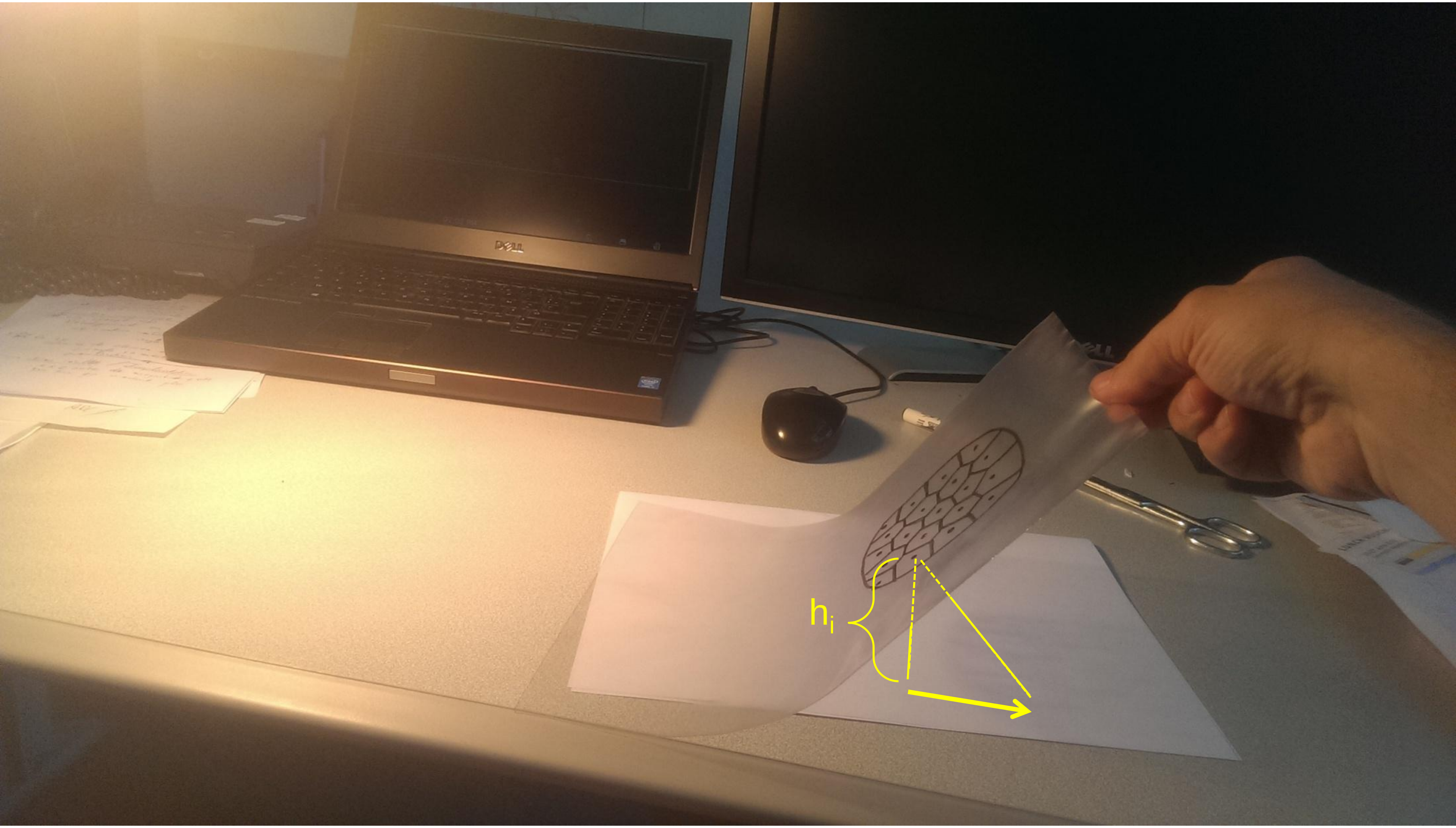
Part. 3 Power Diagrams & Transport



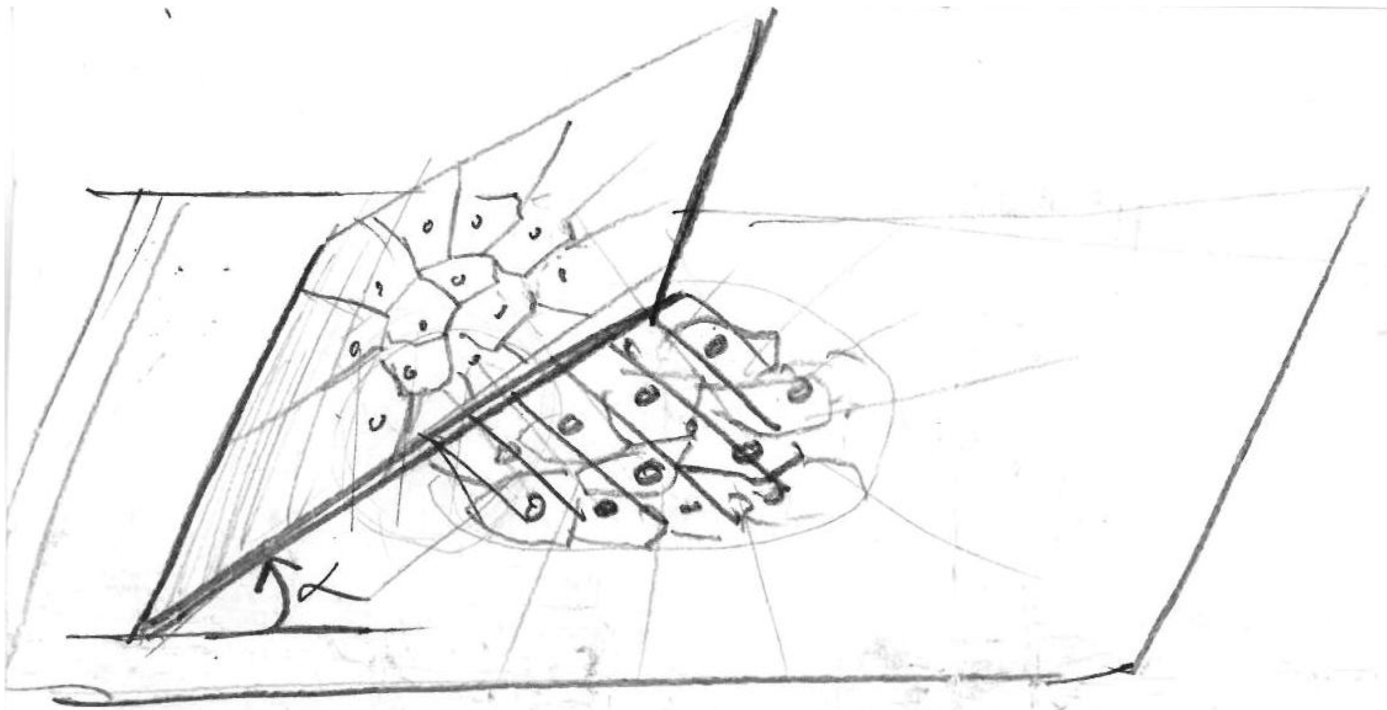
Part. 3 Power Diagrams & Transport



Part. 3 Power Diagrams & Transport



Part. 3 Power Diagrams & Transport

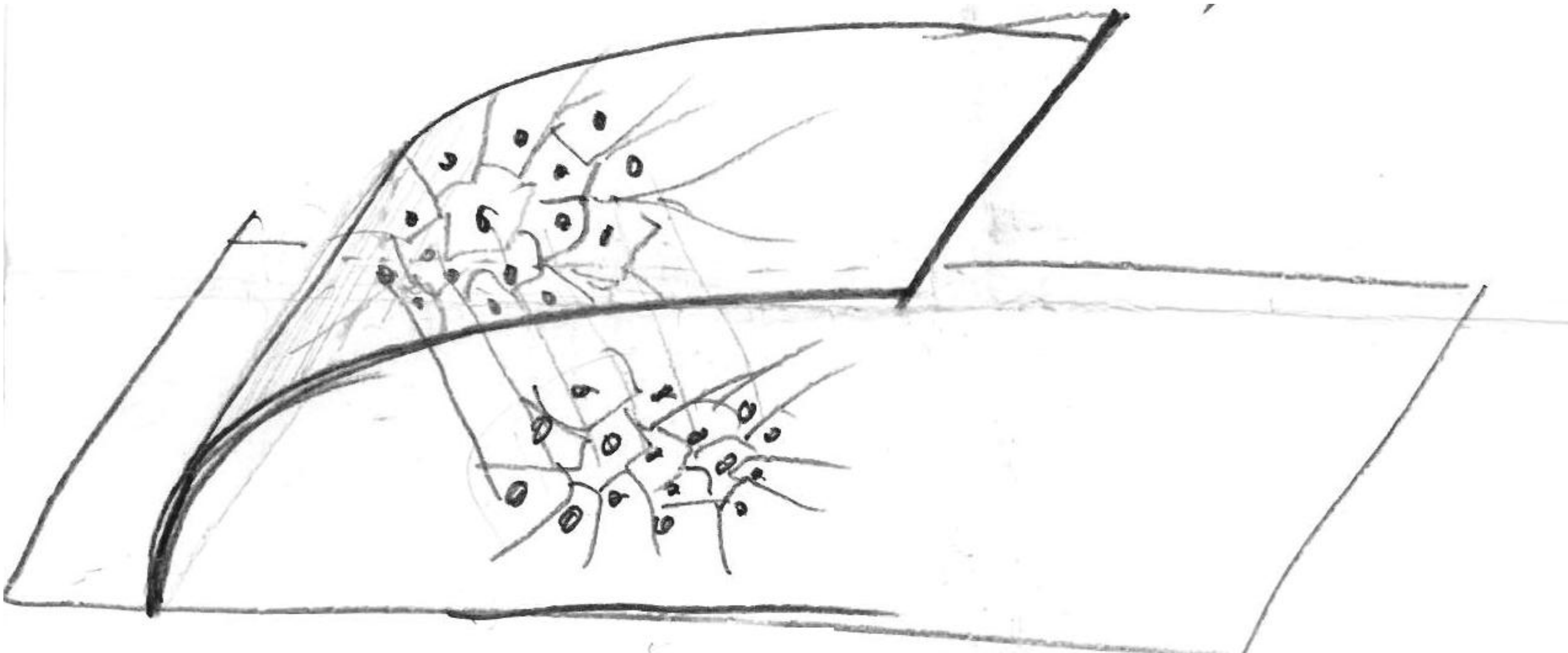


Translating a Voronoi diagram.

1st Try: linear lifting

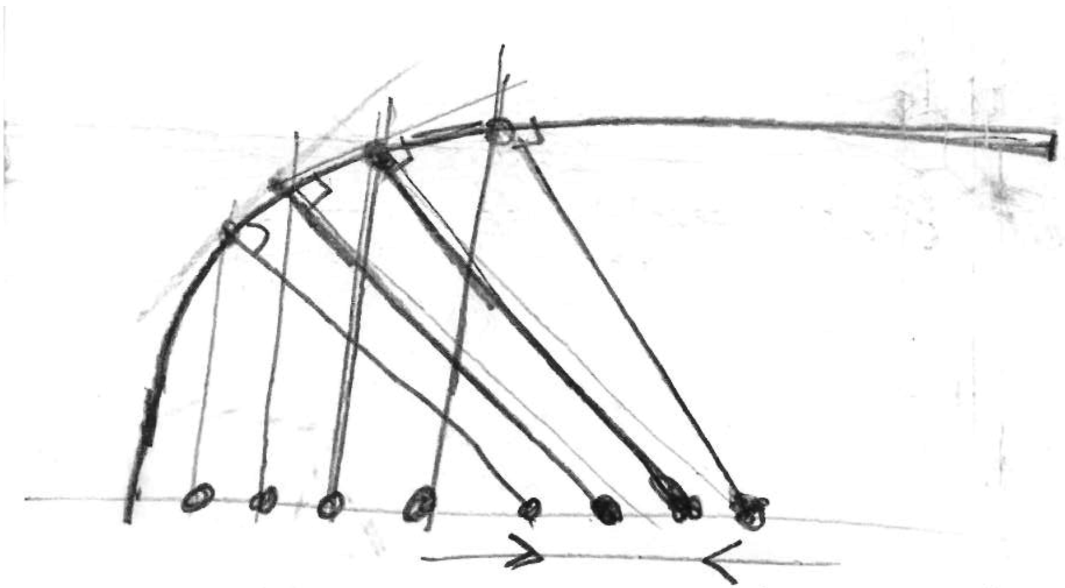
(FAIL: scales by 1/cos(α))

Part. 3 Power Diagrams & Transport



2nd Try : Curved lifting

Part. 3 Power Diagrams & Transport



"converging beams" can compensate the $1/\cos(\alpha)$ expansion by "re-concentrating" the points

Part. 3 Power Diagrams & Transport

(2)

$$d^2(p_i, q) \Big|_{-w_i}^{+h_i^2} < d^2(p_j, q) \overset{+h_i^2}{-w_j} \quad \forall j$$

$$d^2(p_i, q-T) < d^2(p_j, q-T) \quad \forall j$$

$$(p_i - q + T)^2 < (p_j - q + T)^2 \quad \forall j$$

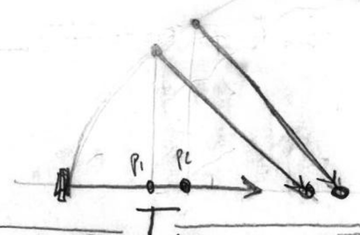
$$d^2(p_i, q) + 2T\cdot(p_i - q) + T^2 < d^2(p_j, q) + 2T(p_j - q) + T^2 \quad \forall j$$

$$d^2(p_i, q) + 2T\cdot p_i < d^2(p_j, q) + 2T\cdot p_j$$

$$w_i \Big|^{h_i^2} = -2T\cdot p_i \quad + cte$$

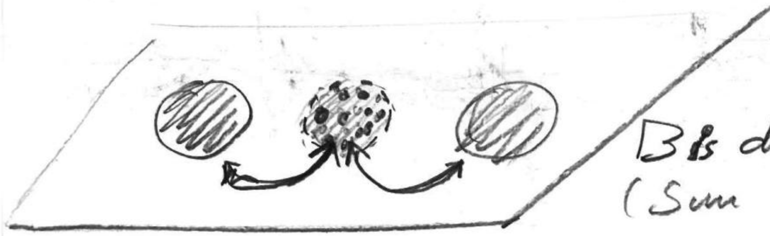
$$h_i^2 = (2T\cdot p_i + Cte)$$

$$\boxed{h_i = \sqrt{2(T\cdot p_i - \min_i(T\cdot p_i))}}$$

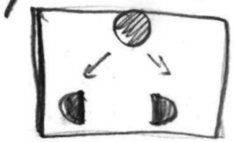


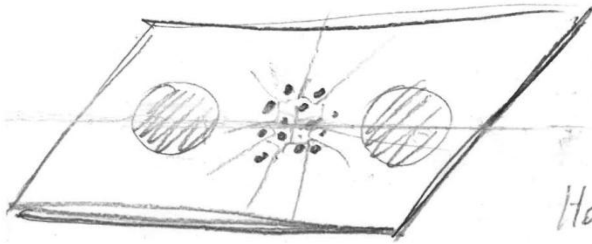
Translation d'un diagramme de Voronoi sectionnel - Relèvement en racine carrée -

Part. 3 Power Diagrams and Transport

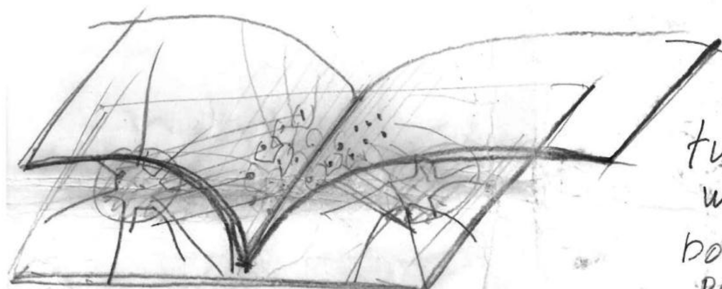


B is discretized (Sum of Diracs)





Voronoi diagram of B samples - How to "back displace" it onto A?



Lifting on two "square root wings" translates both halves of B points into the two blobs of A

Solving for the OTM ($T(x,y)$ vector field) is equivalent to solve for the "square root wings" ($h(x,y)$ scalar function) → Simpler, Unconstrained

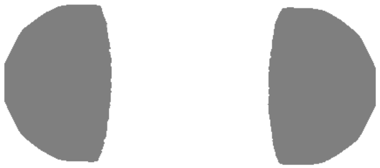
Ref - Name of eqn.

Part. 3 Optimal Transport – 2D example

Numerical Experiment: *Splitting a disk*

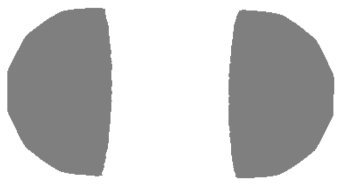
Part. 3 Optimal Transport – 2D example

Numerical Experiment: *Splitting a disk*



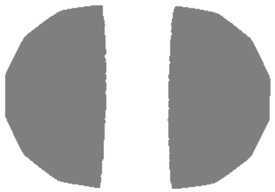
Part. 3 Optimal Transport – 2D example

Numerical Experiment: *Splitting a disk*



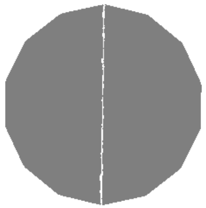
Part. 3 Optimal Transport – 2D example

Numerical Experiment: *Splitting a disk*



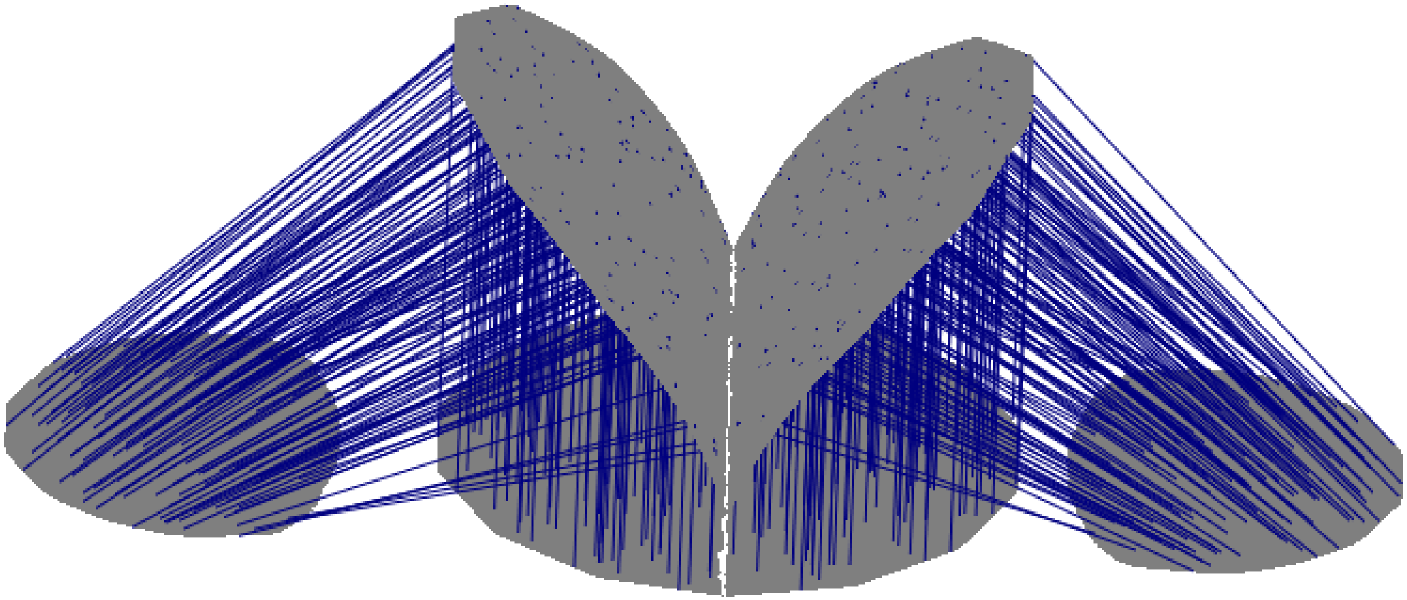
Part. 3 Optimal Transport – 2D example

Numerical Experiment: *Splitting a disk*

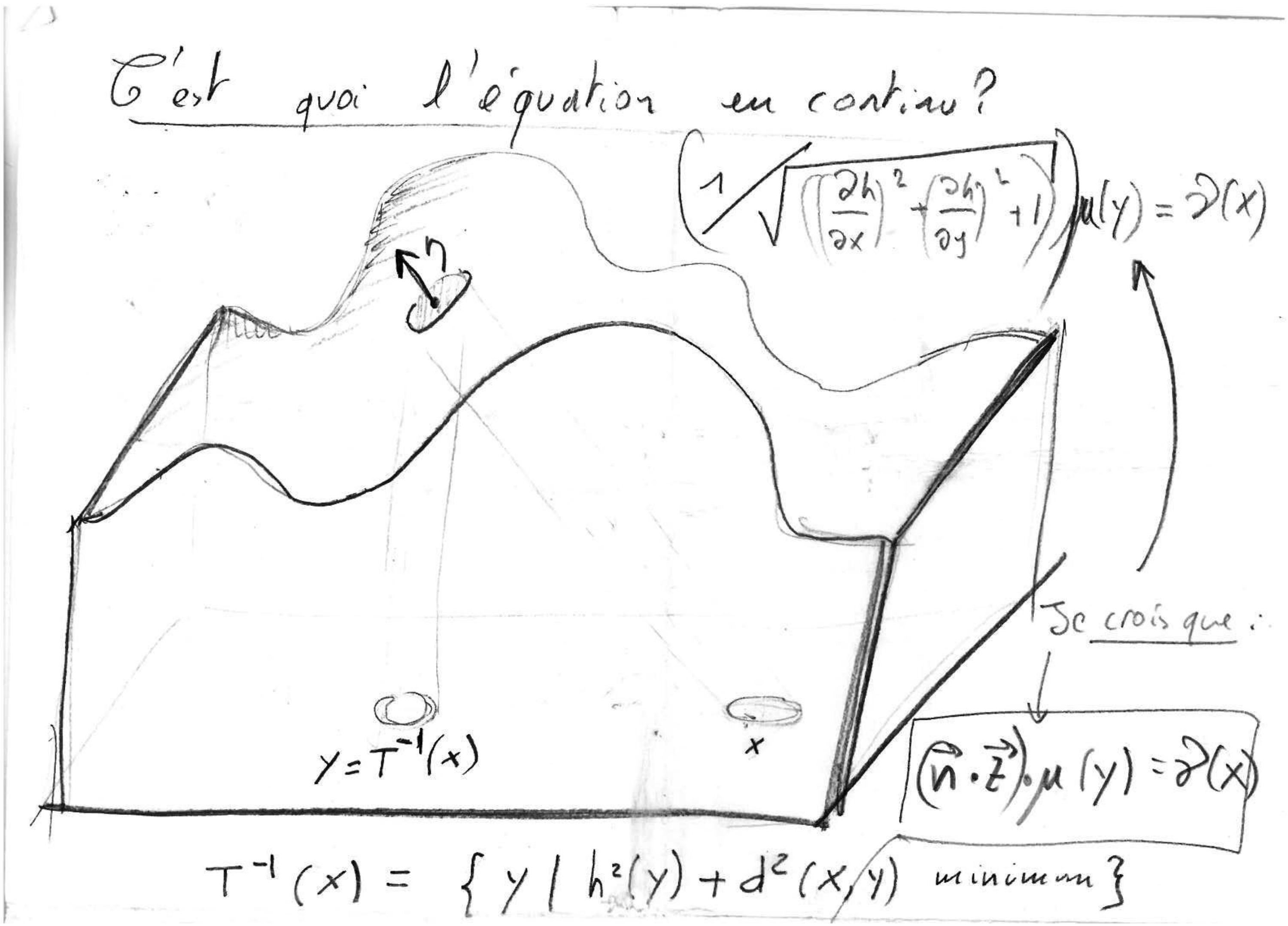


Part. 3 Optimal Transport – 2D example

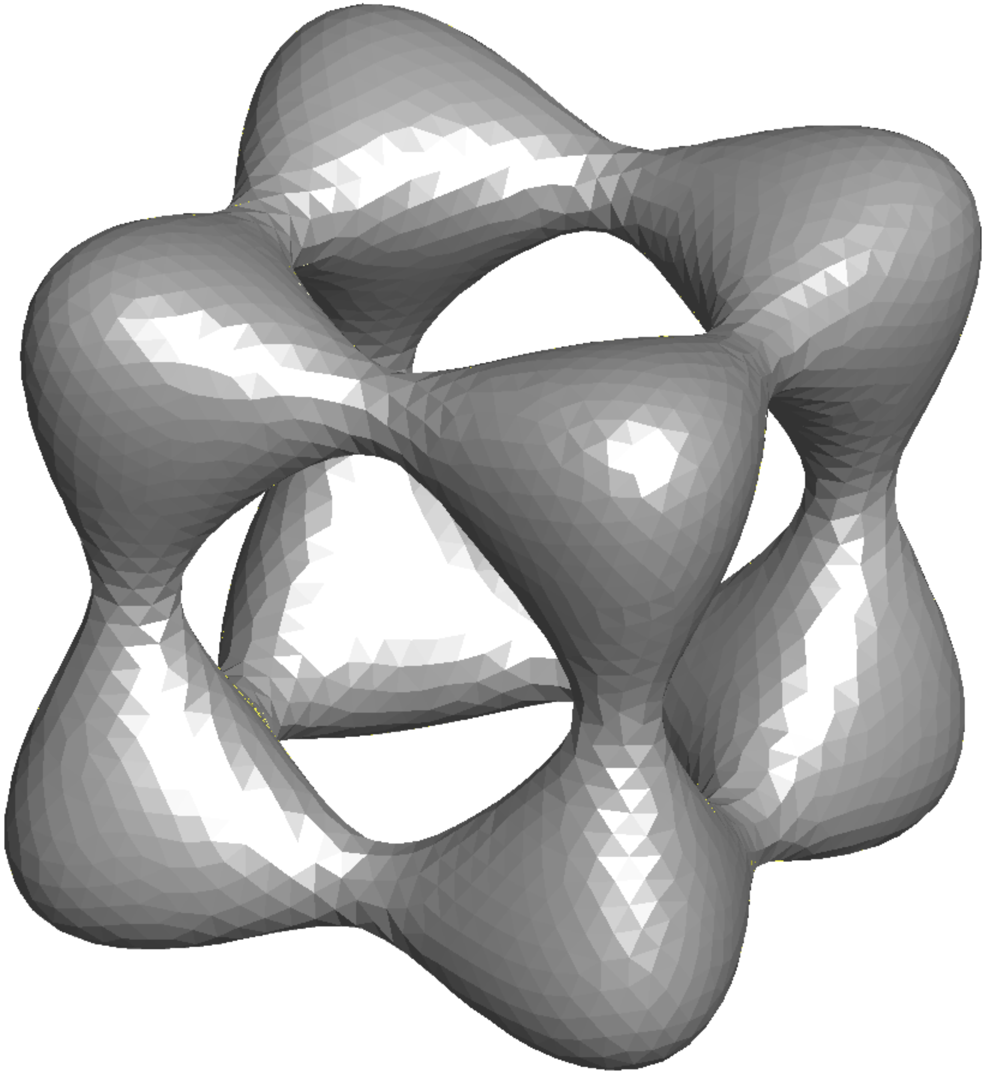
Numerical Experiment: *Splitting a disk*



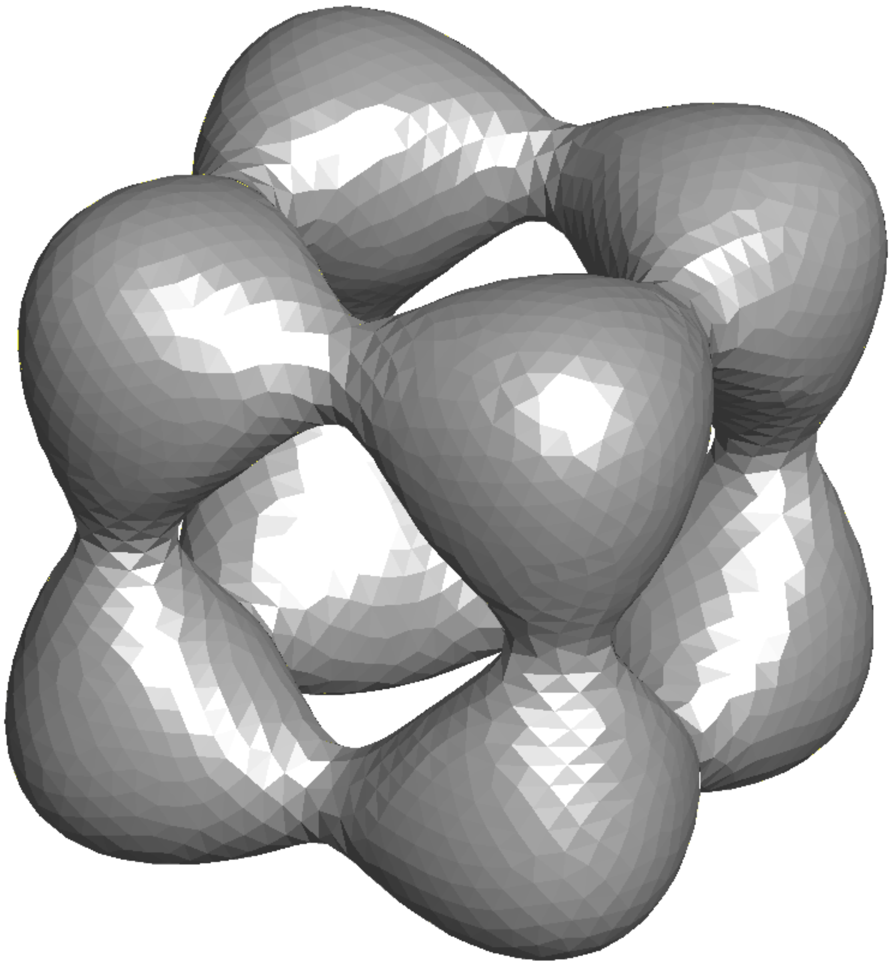
Part. 3 Power Diagrams & Transport



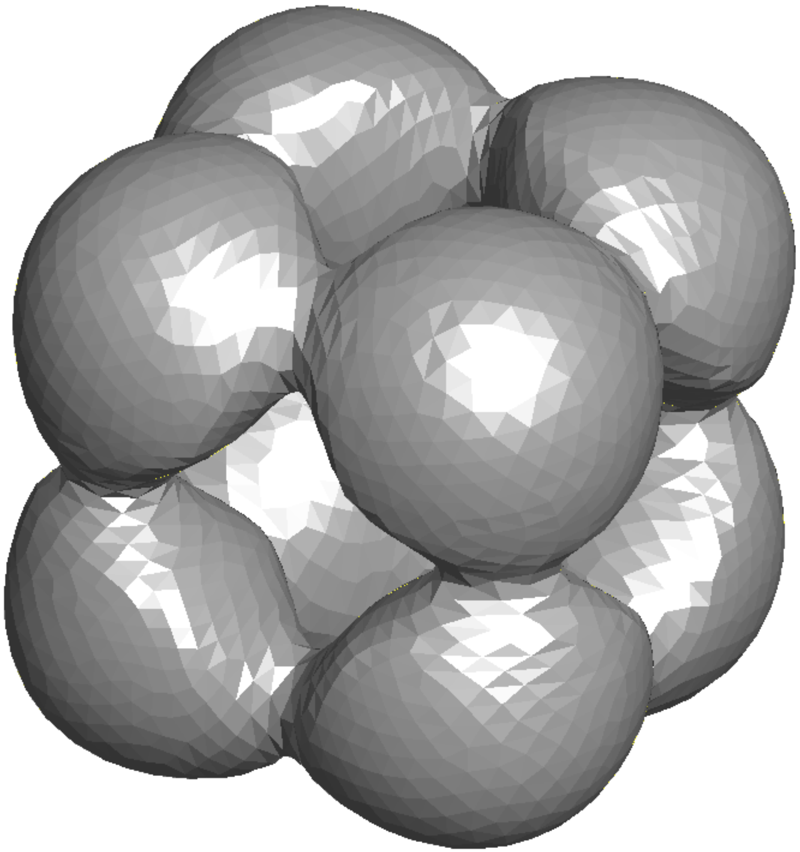
Part. 3 Towards FWD (Fast Wasserstein Distance)



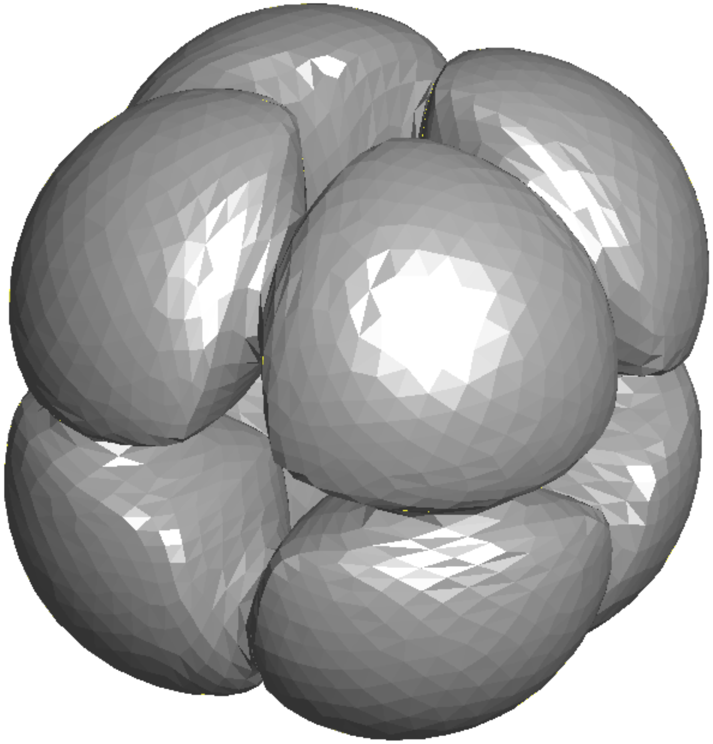
Part. 3 Towards FWD (Fast Wasserstein Distance)



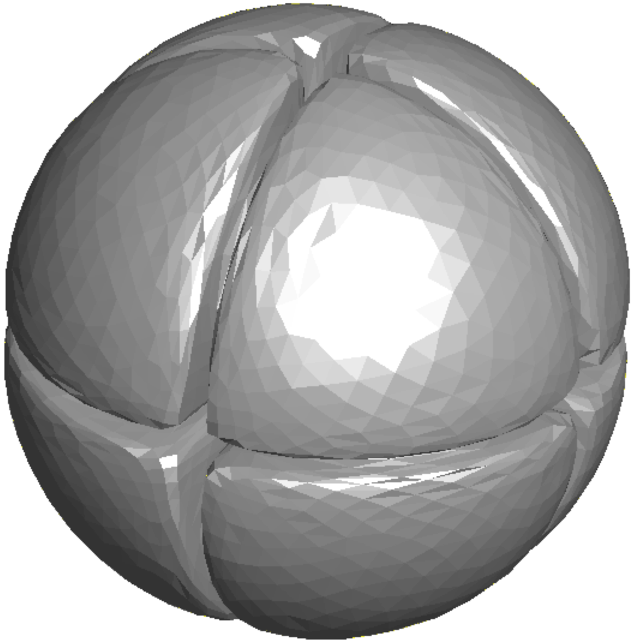
Part. 3 Towards FWD (Fast Wasserstein Distance)



Part. 3 Towards FWD (Fast Wasserstein Distance)



Part. 3 Towards FWD (Fast Wasserstein Distance)



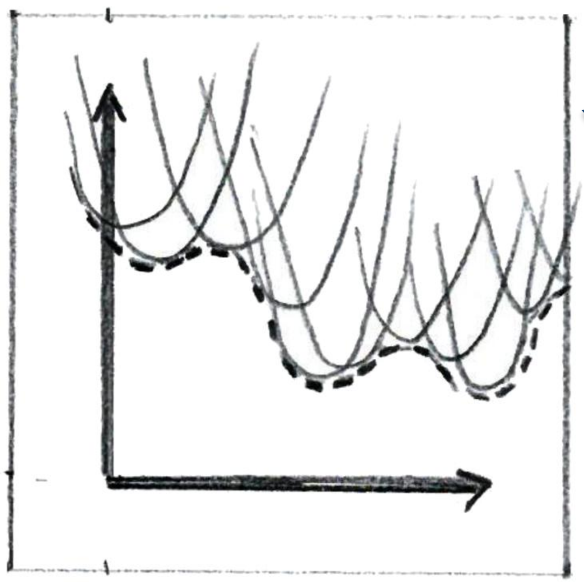
Part. 3 Relation with Vector Quantization

Observation 8. *The quantization noise power $\hat{Q}(\hat{Y})$ computed in $\mathbb{R}^{d+1}$ corresponds to the term $f_{T_W}(W)$ of the function maximized by the weight vector that defines a semi-discrete optimal transport map plus the constant $w_M \mu(\Omega)$.*

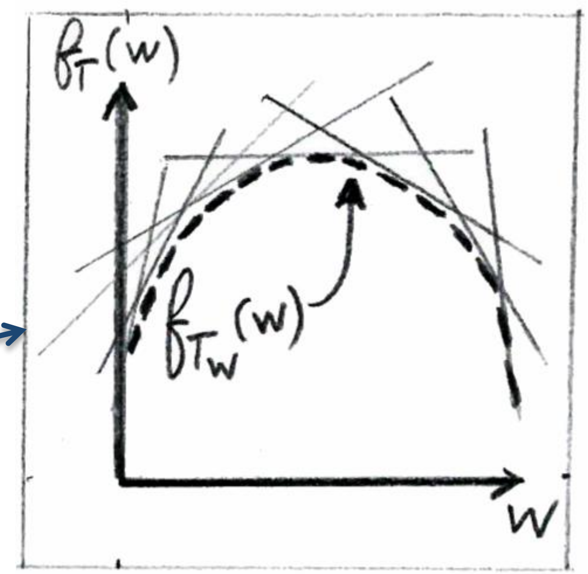
Proof.

$$\begin{aligned}
\hat{Q}(\hat{Y}) &= \sum_i \int_{\mathrm{Vor}(\hat{y}_i)\cap\mathbb{R}^d} \|\hat{x} - \hat{y}_i\|^2 d\mu \\
&= \sum_i \int_{\mathrm{Pow}_W(y_i)} \|x - y_i\|^2 - w_i + w_M d\mu \\
&= f_{T_W}(W) + w_M \mu(\Omega)
\end{aligned}$$

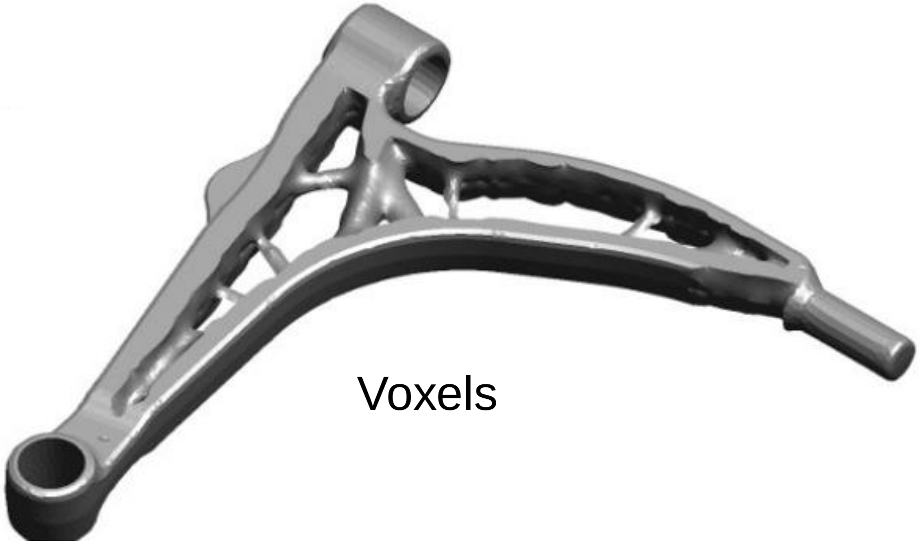
$\square$

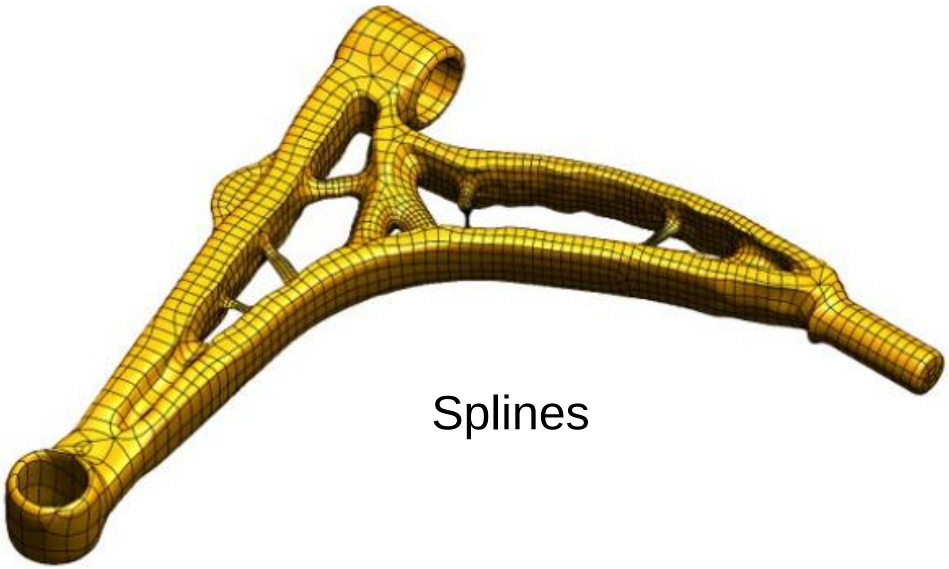


$$\hat{y}_i = \begin{pmatrix} \boxed{\begin{matrix} y_{i,1} \\ y_{i,2} \\ \vdots \\ y_{i,d} \end{matrix}} \\ \boxed{h_i = \sqrt{w_M - w_i}} \end{pmatrix}$$



Part. 3 Self Organizing Optimal Transport Maps





4

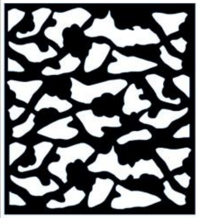
Future Works

Part. 4 Future Works in Fabrication

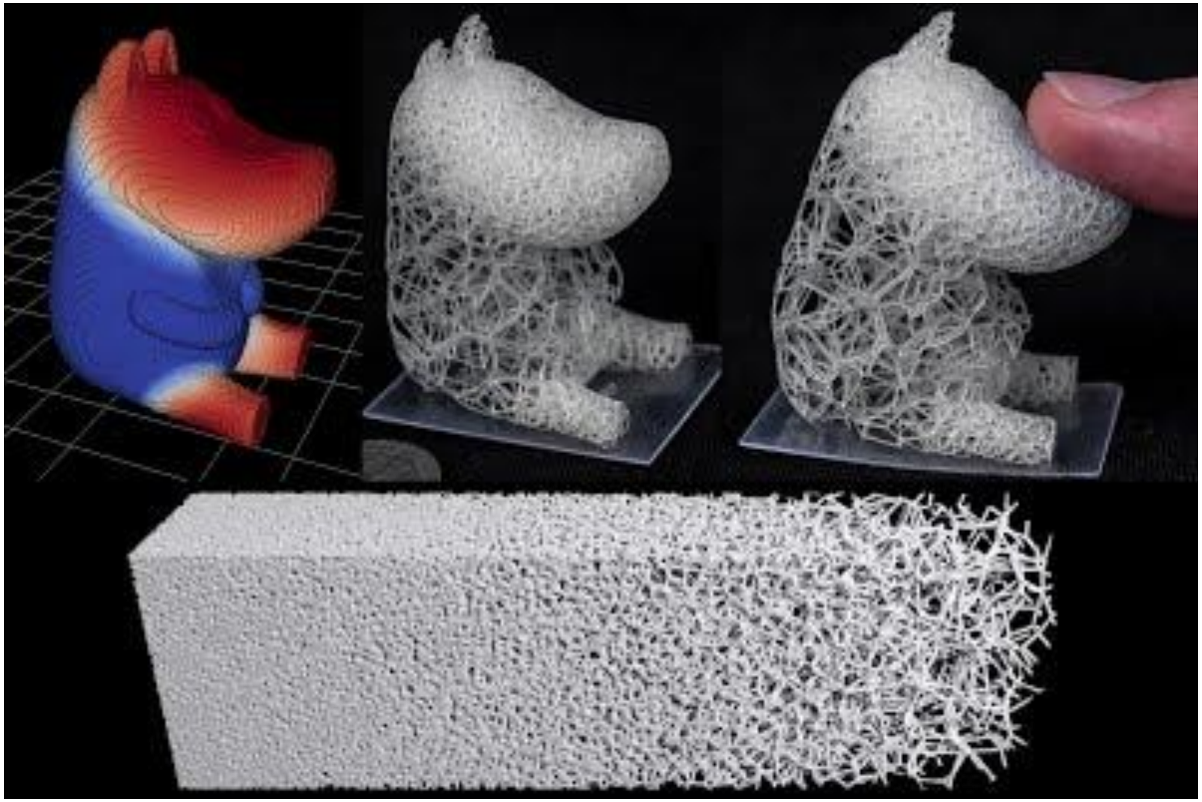
Guiding principles:

(1) Make it easy for everybody !

(2) Integrate more and more fabrication constraints in modeling



Part. 4 Future Works in Fabrication

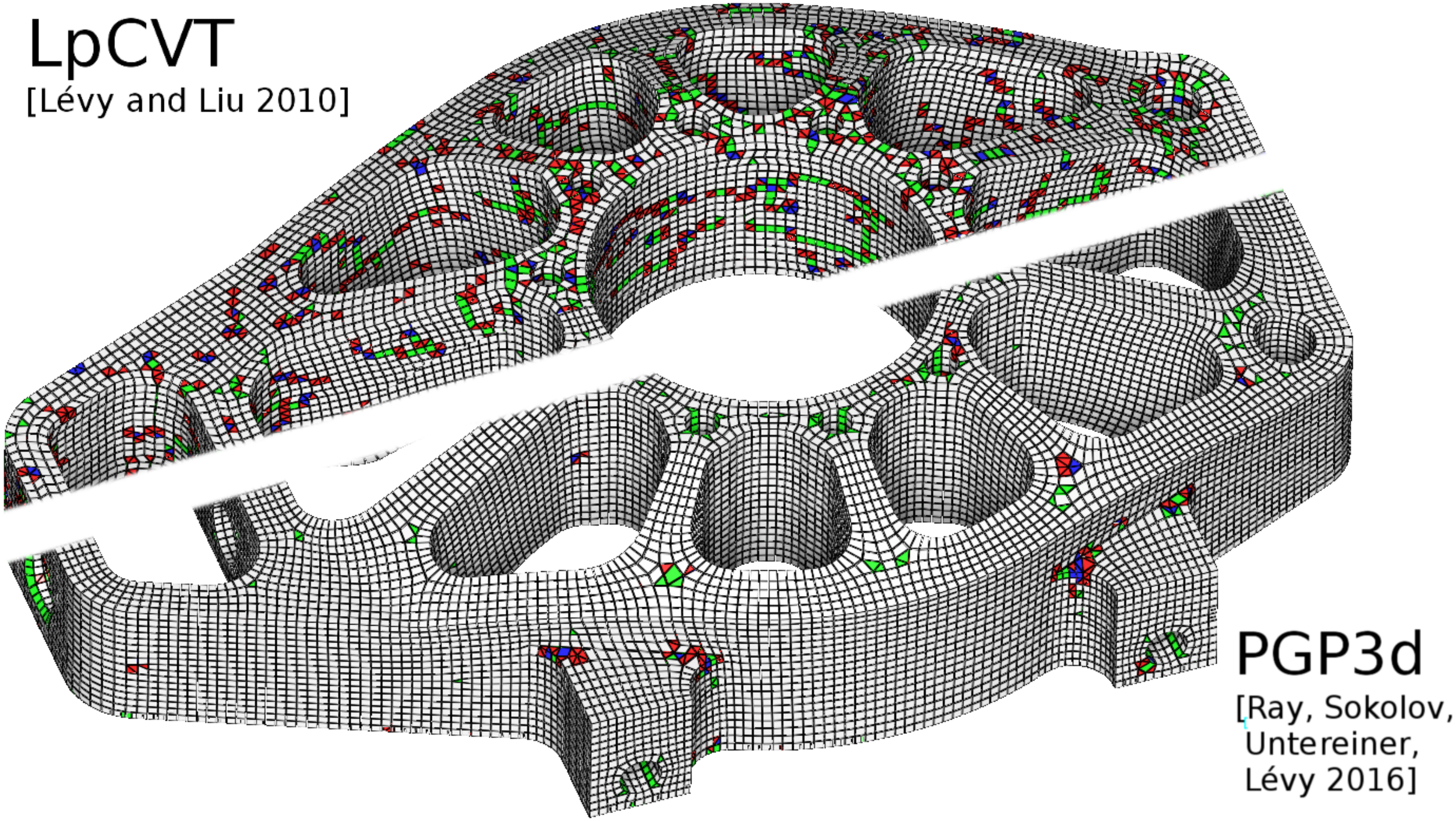


[ACM SIGGRAPH 2016]

Part. 4 Future Works in Applied Mathematics

Discrete Elements – from Equations to Programs

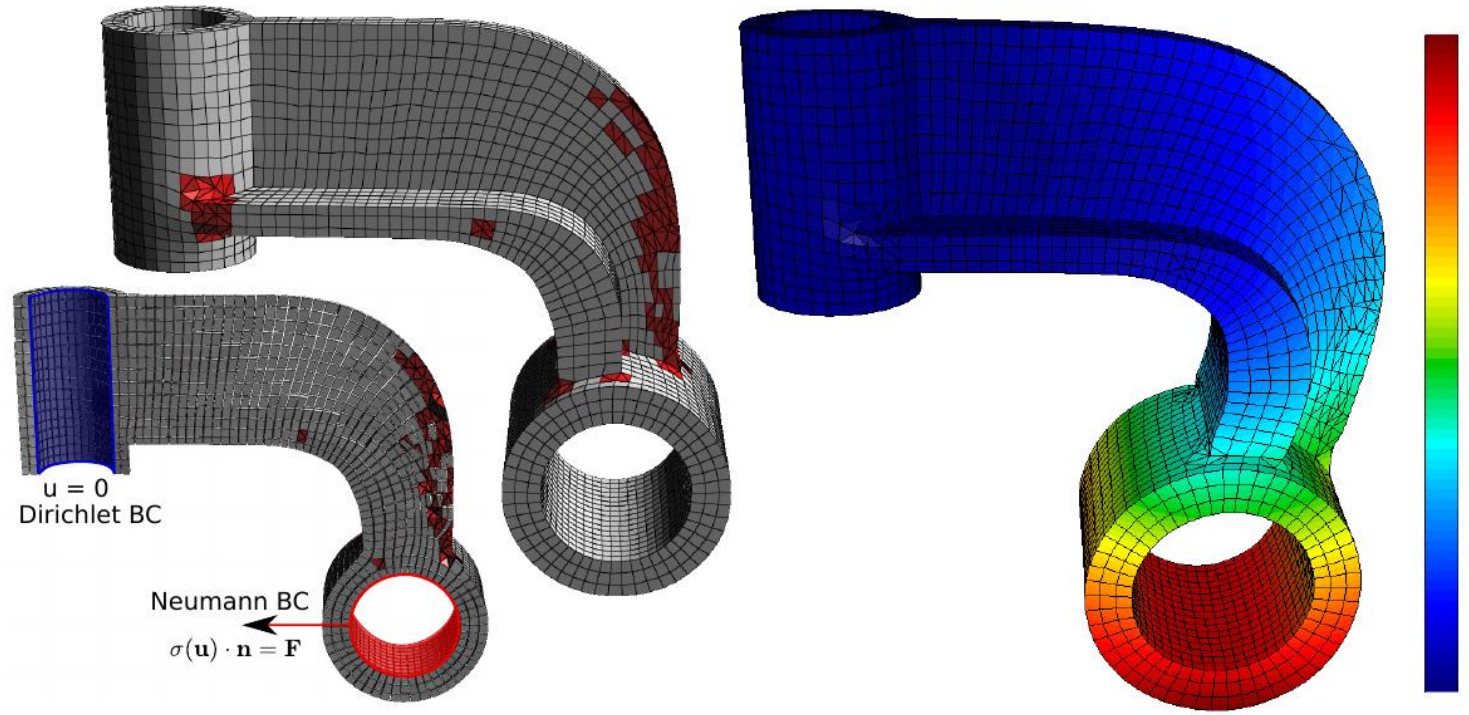
Short term: ***Hex-dominant meshing***



Part. 4 Future Works in Applied Mathematics

Discrete Elements – from Equations to Programs

*Short term: **Hex-dominant meshing***

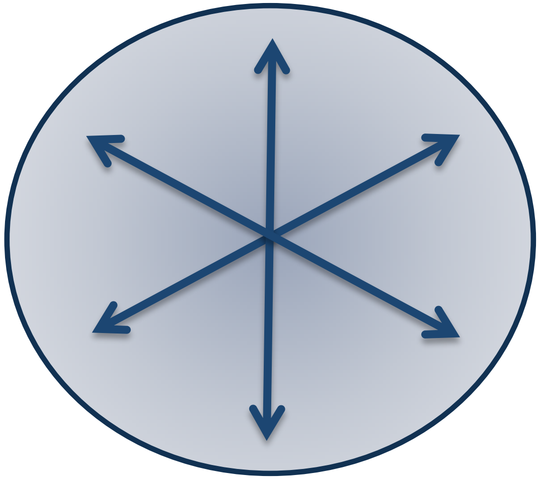


Finite Elements function basis for non-conforming meshes (submitted)

Part. 4 Future Works in Applied Mathematics

Optimization of frame fields for hex-dominant meshing

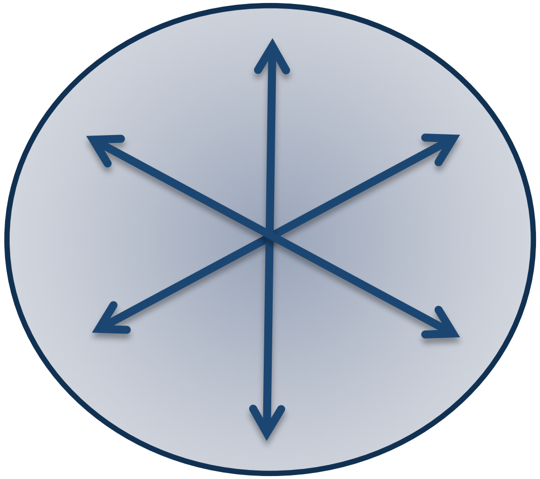
How to interpolate frame fields ?



Part. 4 Future Works in Applied Mathematics

Optimization of frame fields for hex-dominant meshing

How to interpolate frame fields ?



A natural idea:

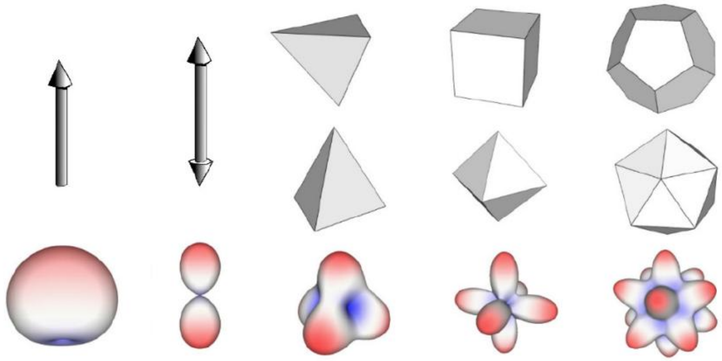
Frame field = 8 Dirac masses on the shere

Optimal Transport for interpolation, barycenters …

Part. 4 Future Works in Applied Mathematics

Optimization of frame fields for hex-dominant meshing

How to interpolate frame fields ?



A natural idea:

Frame field = 8 Dirac masses on the shere

Optimal Transport for interpolation, barycenters …

Not smooth enough

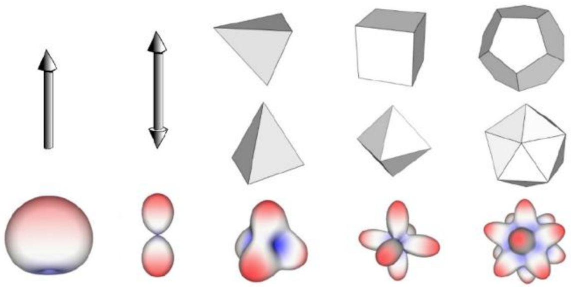
Use “smoothed” version, with functions that has the same symmetries.

Symmetries of platonic solids reproduced with sums of Spherical Harmonics.

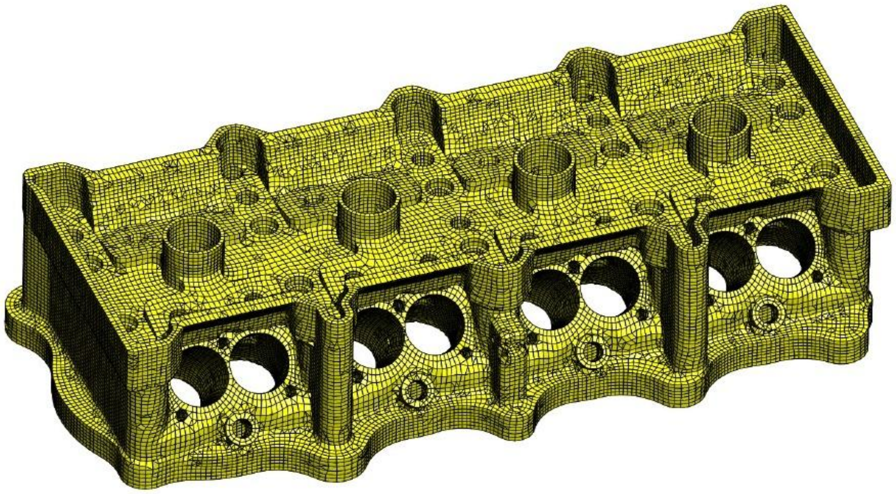
Part. 4 Future Works in Applied Mathematics

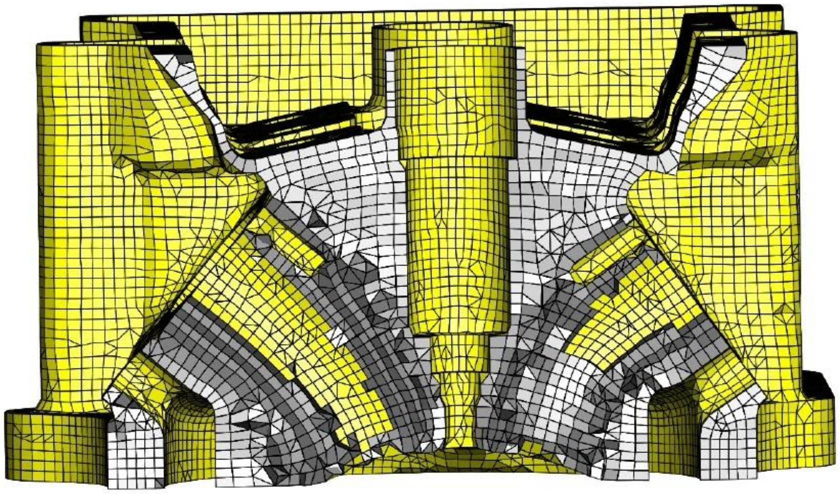
Optimization of frame fields for hex-dominant meshing

How to interpolate frame fields ?



First results are encouraging
(scales-up well)





[ACM Transactions on Graphics 2016]

Part. 4 Future Works in Applied Mathematics

*Longer term**:** from the principle of least action to optimal transport*

JKO scheme (Jordan, Kinderlehrer, Otto)
Benamou, Carlier, Merigot, Oudet arXiv 1408.4536

EXPLORAGRAM project (INRIA exploratory project)
MAGA project (ANR project – submitted)

Part. 4 Future Works in Applied Mathematics

Geometric Predicates: *How can we easily translate* geometric predicates *into computer programs ? How can we* certify *their validity ? Can we invent* programming tools *?*

Source PCK file (using my current version)

```
Sign side2(
    point p0, point p1, point p2,
    point q0, point q1
) {

    scalar l1 = sq_dist(p1,p0) ;
    scalar l2 = sq_dist(p2,p0) ;

    scalar a10 = 2*dot_at(p1,q0,p0);
    scalar a11 = 2*dot_at(p1,q1,p0);
    scalar a20 = 2*dot_at(p2,q0,p0);
    scalar a21 = 2*dot_at(p2,q1,p0);

    scalar Delta = a11 - a10 ;
    scalar DeltaLambda0 = a11 - l1 ;
    scalar DeltaLambda1 =  l1 - a10 ;
    scalar r =
        Delta*l2-a20*DeltaLambda0-a21*DeltaLambda1 ;

    Sign Delta_sign = sign(Delta) ;
    Sign r_sign     = sign(r) ;

    generic_predicate_result(Delta_sign*r_sign) ;

    begin_sos3(p0,p1,p2)
       sos(p0, Sign(Delta_sign*sign(Delta-a21+a20)))
       sos(p1, Sign(Delta_sign*sign(a21-a20)))
       sos(p2, NEGATIVE)
    end_sos
}
```

Source PCK file (using the tools that I plan to develop)

```
Sign side2(point p0, point p1, point p2, point q0, point q1) {

    scalar w0 = 0.0;
    scalar w1 = 0.0;
    scalar w2 = 0.0;

    sos_perturbation(wi, pi, pow(epsilon,i));

    Plane P1 = weighted_bisector(p0,w0,p1,w1);
    Point q = intersection(P1, segment(q0,q1));

    return Sign(sq_dist(q,p0) + w0 - sq_dist(q,p1) - w1);
}
```

sqrt(), root_of() … Voronoi diagram of Segments in 3d doable ?

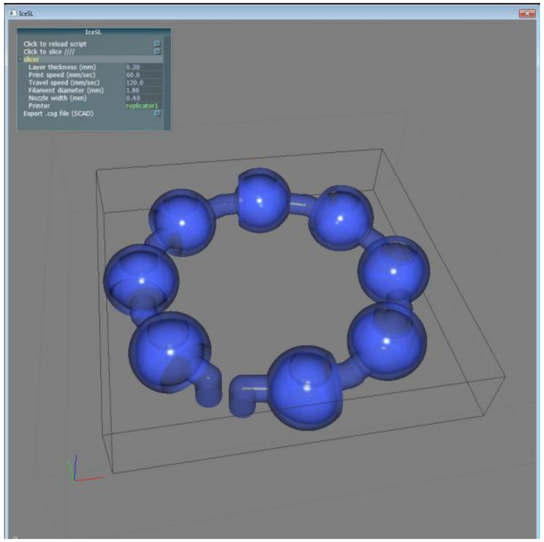
Highlights

ERC StG GOODSHAPE – Optimal Sampling

ERC PoC VORPALINE – Remeshing Software

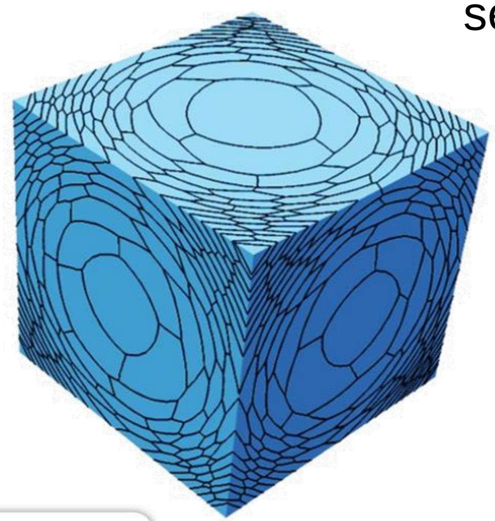
ERC StG SHAPEFORGE – 3D printing made easy

ERC PoC ICEXL – 3D printing – scaling up

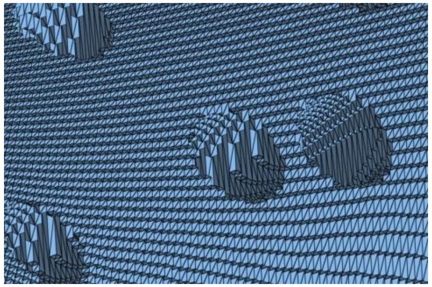


IceSL software – Fast CSG modeler, language, driver for 3d printers …

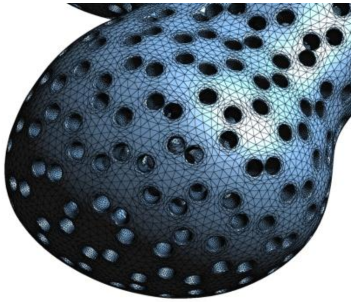
First algorithm that computes aniso. Voro. diagram and
semi-discrete Optimal Transport in 3d (+ Predicate Cons. Kit)



Integration of research results in ALICE !



SHAPEFORGE - Dexels



GOODSHAPE/VORPALINE

Thank you !